MICROLOCALIZATION OF IND-SHEAVES

M. KASHIWARA, P. SCHAPIRA, F. IVORRA, AND I. WASCHKIES

arXiv:math/0407371v1 [math.AG] 22 Jul 2004

Abstract. Let X be a C^∞-manifold and T^*X its cotangent bundle. We construct a microlocalization functor $\mu_X\colon \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X)) \longrightarrow \mathrm{D^b}(\mathrm{I}(\mathbb{K}_{T^*X}))$, where $\mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$ denotes the bounded derived category of ind-sheaves of vector spaces on X over a field $\mathbb{K}$. This functor satisfies $\mathrm{R}\mathcal{H}om(\mu_X(\mathcal{F}), \mu_X(\mathcal{G})) \simeq \mu hom(\mathcal{F}, \mathcal{G})$ for any $\mathcal{F}, \mathcal{G} \in \mathrm{D^b}(\mathbb{K}_X)$, thus generalizing the classical theory of microlocalization. Then we discuss the functoriality of μ_X. The main result is the existence of a microlocal convolution morphism

$$\mu_{X\times Y}(\mathcal{K}_1) \overset{a}{\circ} \mu_{Y\times Z}(\mathcal{K}_2) \longrightarrow \mu_{X\times Z}(\mathcal{K}_1 \circ \mathcal{K}_2)$$

which is an isomorphism under suitable non-characteristic conditions on $\mathcal{K}_1$ and $\mathcal{K}_2$.

Contents

Introduction

This paper is based on ideas of the authors M.K. and P.S. announced in [KS5] and developed in a preliminary manuscript of M.K.

The idea of microlocalization goes back to M. Sato [S] in 1969 who invented the functor of microlocalization of sheaves (along a smooth submanifold of a real manifold) in order to analyze the singularities of hyperfunction solutions of systems of differential equations in the cotangent bundle. This microlocalization procedure then allowed Sato, Kashiwara and Kawai [SKK] to define functorially the sheaf of rings of microdifferential operators

1991 *Mathematics Subject Classification.* Primary:35A27; Secondary:32C38.

Key words and phrases. microlocalization, ind-sheaves.

The author M. K is partially supported by Grant-in-Aid for Scientific Research (B1)13440006, Japan Society for the Promotion of Science and the 21-st century COE program "Formation of an International Center of Excellence in the Frontier of Mathematics and Fostering of Researchers in Future Generations".

The author I. W is partially supported by the same 21-st century COE program .

on the cotangent bundle T^*X of a complex manifold X, a sheaf whose direct image is the sheaf of differential operators on X.

Then in the 80's, M.K. and P.S. (cf. [KS2], [KS3]) developed a microlocal theory of sheaves on a C^∞- manifold X, based on the notion of microsupport (a conic involutive closed subset of the cotangent bundle to X) and introduced in particular the functor μhom. This is roughly speaking a functor which associate to a pair of sheaves on X the sheaf of microlocal morphisms between them.

On the other hand, the Riemann-Hilbert problem, solved by M.K., tells us that there is a one-to-one correspondence between the regular holonomic modules over the ring of differential operators and the perverse sheaves. The notion of regular holonomic modules over the ring of differential operators can be easily microlocalized to the notion of regular holonomic modules over the ring of microdifferential operators and it is a natural question to ask if there is a natural notion of microlocalization of perverse sheaves, or, more generally a functor μ of microlocalization for sheaves, the microsupport of a sheaf being the support of its microlocalization and the functor μhom being the internal *hom* applied to the microlocalization. This is indeed what we do in this paper.

As an application of the new functor μ, the author I.W. [W] has recently constructed the stack of microlocal perverse sheaves on a homogeneous symplectic manifold, after M.K. [K] had constructed the stack of microdifferential modules.

The paper consists of two parts. The first part is the technical heart of the paper. We define kernels on a C^∞-manifold X, attached to the data of a closed submanifold Z and a 1-form σ vanishing on Z. Then we study its functorial properties. These kernels can be seen as "general" microlocalization kernels, though their only role in this paper is to provide us with the tools for the proofs of the functorial properties of μ.

In the second part we introduce the functor μ, which is the integral transform with respect to the kernel K_{T^*X} on $T^*X \times T^*X$ associated with the fundamental 1-form. We discuss the functorial properties of μ deduced from the corresponding properties of the kernels studied in the first part. We then show how some classical microlocal properties can be generalized to ind-sheaves. We give a comparison theorem between the micro-support of ind-sheaves $\mathcal{F}$ and the support of its microlocalization $\mu(\mathcal{F})$.

As an application, we prove that, on a complex manifold X, μhom induces a well-defined functor

$$\mu hom(\,\bullet\,, \mathcal{O}_X)\colon\ \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)^{\mathrm{op}} \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathcal{E}_X),$$

where $\mathcal{E}_X$ is the ring of microdifferential operators.

The authors would like to thank A. D'Agnolo for many helpful comments.

1. Microlocal kernels

In all this paper, $\mathbb{K}$ denotes a field.

1.1. **Review on Ind-sheaves on manifolds.**

In this section we shall give a short overview on the theory of ind-sheaves of [KS1].

Let X be a locally compact topological space with finite cohomological dimension, $\mathrm{Mod}(\mathbb{K}_X)$ the category of sheaves of $\mathbb{K}$-vector spaces on X, and $\mathrm{Mod}^{\mathrm{c}}(\mathbb{K}_X)$ its full subcategory of sheaves with compact supports.

We denote by $\mathrm{I}(\mathbb{K}_X)$ the category of ind-sheaves, which is by definition the category of ind-objects of $\mathrm{Mod}^{\mathrm{c}}(\mathbb{K}_X)$. Then, $\mathrm{I}(\mathbb{K}_X)$ is an abelian category, and its bounded derived category is denoted by $\mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$.

There is a fully faithful exact functor

$$\iota_X\colon \operatorname{Mod}(\mathbb{K}_X) \longrightarrow \mathrm{I}(\mathbb{K}_X) \quad \text{given by} \quad F \mapsto \text{“}\varinjlim_{U\subset\subset X}\text{”}\, F_U,$$

where the direct limit on the right is taken over the family of relatively compact open subsets U of X. In the sequel, we will regard $\operatorname{Mod}(\mathbb{K}_X)$ as a full subcategory of $\mathrm{I}(\mathbb{K}_X)$.

The functor ι_X admits an exact left adjoint functor

$$\alpha_X\colon \mathrm{I}(\mathbb{K}_X) \longrightarrow \operatorname{Mod}(\mathbb{K}_X), \quad \text{“}\varinjlim_{i\in I}\text{”}\, F_i \mapsto \varinjlim_{i\in I} F_i.$$

Since ι_X is fully faithful, we have $\alpha_X \circ \iota_X \simeq \mathrm{Id}_{\operatorname{Mod}(\mathbb{K}_X)}$.
The functor α_X admits an exact fully faithful left adjoint

$$\beta_X\colon \operatorname{Mod}(\mathbb{K}_X) \longrightarrow \mathrm{I}(\mathbb{K}_X).$$

Since β_X is fully faithful, we get $\alpha_X \circ \beta_X \simeq \mathrm{Id}_{\operatorname{Mod}(\mathbb{K}_X)}$. The functor β_X is less easy to define than α_X and ι_X. However, for a locally closed subset $S \subset X$,

$$\widetilde{\mathbb{K}}_S := \beta_X(\mathbb{K}_S)$$

is described as follows. Let Z be a closed subset, then we have

$$\widetilde{\mathbb{K}}_Z \simeq \text{“}\varinjlim_{Z\subset W}\text{”}\, \mathbb{K}_{\overline{W}},$$

where W runs through the open subsets containing Z. If $U \subset X$ is an open subset then

$$\widetilde{\mathbb{K}}_U \simeq \text{“}\varinjlim_{V\subset\subset U}\text{”}\, \mathbb{K}_V,$$

where V runs through the family of relatively compact open subsets of U. If $S \subset X$ is locally closed, then we can write $S = Z \cap U$ where U is open and Z is closed, and

$$\widetilde{\mathbb{K}}_S \simeq \widetilde{\mathbb{K}}_U \otimes \widetilde{\mathbb{K}}_W \simeq \text{“}\varinjlim_{V\subset\subset U,\, Z\subset W}\text{”}\, \mathbb{K}_{V\cap\overline{W}}\,.$$

Therefore $\mathbb{K}_{V\cap\overline{W}} \to \mathbb{K}_S$ induces a morphism $\widetilde{\mathbb{K}}_S \longrightarrow \mathbb{K}_S$ which is not an isomorphism in general.

Note that if Z is closed and $S \subset Z$ is a locally closed subset, then

$$\mathbb{K}_S \otimes \widetilde{\mathbb{K}}_Z \simeq \mathbb{K}_S\,.$$

The machinery of Grothendieck's six operations is also applied to this context. We have the functors:

$$\begin{aligned}
f^{-1},\, f^! \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_Y)) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)),\\
\mathrm{R}f_*,\, \mathrm{R}f_{!!} \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_Y)),\\
\mathrm{R}\mathcal{IH}om \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)) \to \mathrm{D}^{+}(\mathrm{I}(\mathbb{K}_X)),\\
\otimes \quad &: \quad \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)) \times \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)),
\end{aligned}$$

(here, $f\colon X \to Y$ is a continuous map) and we have the stack-theoretical hom

$$\mathrm{R}\mathcal{H}om\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)) \to \mathrm{D}^{+}(\mathbb{K}_X).$$

Note that the functor $\mathrm{R}\mathcal{IH}om$ sends $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$ to $\mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$ and $\mathrm{R}\mathcal{H}om$ sends $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)^{\mathrm{op}} \times \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$ to $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$.

The inverse image functor f^{-1} is a left adjoint of the direct image functor $\mathrm{R}f_*$. The functor of direct image with proper support $\mathrm{R}f_{!!}$ has a right adjoint functor $f^!$. Most formulas of sheaves have their counterpart in the theory of ind-sheaves, but some formulas are new. We shall not repeat them here and refer to [KS1]. As an example we state the following propositions:

Proposition 1.1.1. *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Then we have canonical isomorphisms

$$\mathrm{R}f'_{!!}g'^{-1} \simeq g^{-1}\,\mathrm{R}f_{!!}, \qquad \mathrm{R}f'_*g'^! \simeq g^!\,\mathrm{R}f_*, \qquad \mathrm{R}f'_{!!}g'^! \simeq g^!\,\mathrm{R}f_{!!}.$$

Note that the last isomorphism has no counterpart in sheaf theory.

Proposition 1.1.2. *For a morphism $f\colon X \to Y$ and for $K \in \mathrm{D^b}(\mathbb{K}_Y)$, $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$, we have*

$$\begin{aligned} &\mathrm{R}f_{!!}\,\mathrm{R}\mathcal{I}hom(f^{-1}K, \mathcal{F}) \simeq \mathrm{R}\mathcal{I}hom(K, \mathrm{R}f_{!!}\mathcal{F}) \quad \text{in } \mathrm{D^b}(\mathrm{I}(\mathbb{K}_Y)),\\ &\mathrm{R}f_!\,\mathrm{R}\mathcal{H}om(f^{-1}K, \mathcal{F}) \simeq \mathrm{R}\mathcal{H}om(K, \mathrm{R}f_{!!}\mathcal{F}) \qquad \text{in } \mathrm{D^b}(\mathbb{K}_Y).\end{aligned}$$

Remark 1.1.3. Let Z be a closed subset of X and let $i\colon Z \to X$, $j\colon X \setminus Z \to Z$ be the inclusion morphisms. Then for $\mathcal{F}$, $\mathcal{F}' \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$, we have

$$\begin{aligned} &\mathrm{R}j_{!!}j^{-1}\mathcal{F} \simeq \widetilde{\mathbb{K}}_{X\setminus Z} \otimes \mathcal{F}, && \mathrm{R}i_*i^{-1}\mathcal{F} \simeq \mathbb{K}_Z \otimes \mathcal{F},\\ &\mathrm{R}j_*j^{-1}\mathcal{F} \simeq \mathrm{R}\mathcal{I}hom(\widetilde{\mathbb{K}}_{X\setminus Z}, \mathcal{F}), && \mathrm{R}i_*i^!\mathcal{F} \simeq \mathrm{R}\mathcal{I}hom(\mathbb{K}_Z, \mathcal{F}),\\ &\mathrm{R}j_*j^{-1}\,\mathrm{R}\mathcal{H}om(\mathcal{F}', \mathcal{F}) \simeq \mathrm{R}\mathcal{H}om(\widetilde{\mathbb{K}}_{X\setminus Z} \otimes \mathcal{F}', \mathcal{F}). \end{aligned} \tag{1.1}$$

Hence there are *not* distinguished triangles

$$\mathrm{R}j_{!!}j^{-1}\mathcal{F} \to \mathcal{F} \to \mathrm{R}i_*i^{-1}\mathcal{F} \xrightarrow{+1} \text{ nor } \mathrm{R}i_*i^!\mathcal{F} \to \mathcal{F} \to \mathrm{R}j_*j^{-1}\mathcal{F} \xrightarrow{+1},$$

and instead there are distinguished triangles

$$\mathrm{R}j_{!!}j^{-1}\mathcal{F} \to \mathcal{F} \to \mathcal{F} \otimes \widetilde{\mathbb{K}}_Z \xrightarrow{+1} \text{ and } \mathrm{R}\mathcal{I}hom(\widetilde{\mathbb{K}}_Z, \mathcal{F}) \to \mathcal{F} \to \mathrm{R}j_*j^{-1}\mathcal{F} \xrightarrow{+1}. \tag{1.2}$$

The functor β satisfies the following properties.

$$\beta_X(F) \otimes \beta_X(G) \simeq \beta_X(F \otimes G) \text{ for } F,\, G \in \mathrm{D^b}(\mathbb{K}_X). \tag{1.3 a}$$

For $f\colon X \to Y$ and $G \in \mathrm{D^b}(\mathbb{K}_Y)$ and $\mathcal{G} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$, we have

$$f^{-1}\beta_Y(G) \simeq \beta_X(f^{-1}G) \quad \text{and} \quad f^!(\mathcal{G} \otimes \beta_Y(G)) \simeq f^!\mathcal{G} \otimes \beta_X(f^{-1}G). \tag{1.3 b}$$

For $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$ and K, $K' \in \mathrm{D^b}(\mathbb{K}_X)$, we have

$$\begin{aligned} \mathrm{R}\mathcal{I}hom(K, \mathcal{F}) \otimes \beta_X(K') &\simeq \mathrm{R}\mathcal{I}hom\left(K, \mathcal{F} \otimes \beta_X(K')\right) \quad \text{in } \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X)),\\ \mathrm{R}\mathcal{H}om(K, \mathcal{F}) \otimes K' &\simeq \mathrm{R}\mathcal{H}om\left(K, \mathcal{F} \otimes \beta_X(K')\right) \quad \text{in } \mathrm{D^b}(\mathbb{K}_X). \end{aligned} \tag{1.3 c}$$

In general β does not commute with direct image.

Lemma 1.1.4. *Consider a closed embedding $i\colon Z \hookrightarrow X$ and $F \in \mathrm{D^b}(\mathbb{K}_Z)$. Then we have an isomorphism*

$$\beta_X(\mathrm{R}i_*F) \otimes \mathbb{K}_Z \simeq \mathrm{R}i_*\beta_Z(F).$$

Proof. We have

$$\beta_X(\mathrm{R}i_*F)\otimes\mathbb{K}_Z\simeq\mathrm{R}i_*i^{-1}\beta_X(\mathrm{R}i_*F)\simeq\mathrm{R}i_*\beta_Z(i^{-1}\,\mathrm{R}i_*F)\simeq\mathrm{R}i_*\beta_Z(F).$$

□

The following fact will be used frequently in the paper:

(1.4) A morphism $u\colon\mathcal{F}\to\mathcal{G}$ in $\mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$ is an isomorphism if and only if $\mathcal{F}\otimes\widetilde{\mathbb{K}}_x\to\mathcal{G}\otimes\widetilde{\mathbb{K}}_x$ is an isomorphism for all $x\in X$.

We list the commutativity of various functors. Here, "○" means that the functors commute, and "×" that they do not.

	ι	α	β	$\varinjlim$
$\otimes$	○	○	○	○
f^{-1}	○	○	○	○
$\mathrm{R}f_*$	○	○	×	×
$\mathrm{R}f_{!!}$	×	○	×	○
$f^!$	○	×	×	○
$\varinjlim$	×	○	○	

In the table, $\varinjlim$ means filtrant inductive limits. For example, the commutativity of $\mathrm{R}f_{!!}$ and $\varinjlim$ should be understood as in Proposition 2.3.2 (i) below.

Notation 1.1.5. For a continuous map $f\colon X\to Y$, we denote by $\omega_{X/Y}$ the topological dualizing sheaf $f^!\,\mathbb{K}_Y$, and $\omega_X=\omega_{X/\{\mathrm{pt}\}}$. If X and Y are manifolds, $\omega_{X/Y}\simeq\omega_X\otimes f^{-1}\omega_Y^{\otimes-1}$.

For three manifolds X_i $(i=1,2,3)$ and for kernels $K\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_1\times X_2}))$ and $K'\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_2\times X_3}))$, we define their convolution by

$$K\underset{X_2}{\circ}K'=\mathrm{R}p_{13!!}(p_{12}^{-1}K\otimes p_{23}^{-1}K'),\tag{1.5}$$

where p_{ij} is the projection from $X_1\times X_2\times X_3$ to $X_i\times X_j$. We sometimes denote it simply by $K\circ K'$ when there is no risk of confusion. This product of kernels satisfies the associative law:

$$(K\circ K')\circ K''\simeq K\circ(K'\circ K'')$$

for $K\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_1\times X_2}))$, $K'\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_2\times X_3}))$ and $K''\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_3\times X_4}))$. By taking $\{\mathrm{pt}\}$ as X_3 in (1.5), we obtain the integral transform functor:

$$K\circ\;:\;\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_2}))\to\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_1})).$$

The following lemma is frequently used in §2.

Lemma 1.1.6. *Let* $f_k\colon X_k\to Y_k$ $(k=1,2,3)$ *be morphisms and* $\mathcal{K}_{ij}\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_i\times X_j}))$ *and* $\mathcal{L}_{ij}\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{Y_i\times Y_j}))$.

(i) $\big((f_1\times\mathrm{id}_{Y_2})^{-1}\mathcal{L}_{12}\big)\underset{Y_2}{\circ}\big((\mathrm{id}_{Y_2}\times f_3)^{-1}\mathcal{L}_{23}\big)\simeq(f_1\times f_3)^{-1}(\mathcal{L}_{12}\underset{Y_2}{\circ}\mathcal{L}_{23})$ *in* $\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X_1\times X_3}))$,

(ii) $\big((f_1\times\mathrm{id}_{X_2})_{!!}\mathcal{K}_{12}\big)\underset{Y_2}{\circ}\big((\mathrm{id}_{X_2}\times f_3)_{!!}\mathcal{K}_{23}\big)\simeq(f_1\times f_3)_{!!}(\mathcal{K}_{12}\underset{X_2}{\circ}\mathcal{K}_{23})$ *in* $\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{Y_1\times Y_3}))$,

(iii) $\big((\mathrm{id}_{Y_1}\times f_2)^{-1}\mathcal{L}_{12}\big)\underset{X_2}{\circ}\mathcal{K}_{23}\simeq\mathcal{L}_{12}\underset{Y_2}{\circ}\mathrm{R}(f_2\times\mathrm{id}_{X_3})_{!!}\mathcal{K}_{23}$ *in* $\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{Y_1\times X_3}))$.

1.2. **Kernels attached to 1-forms.** Let us denote by $\pi_X\colon T^*X \to X$ the cotangent bundle to X. For a closed submanifold Z of X, we denote by T^*_ZX its conormal bundle. In particular, T^*_XX is the zero section of T^*X. To a differentiable map $f\colon X \to Y$, we associate the diagram

$$T^*X \xleftarrow{f_d} T^*Y \times_Y X \xrightarrow{f_\pi} T^*Y.$$

Notation 1.2.1. For a vector bundle $p\colon E \to X$, we denote by $\dot{E}$ the space E with the zero section removed, and by $\dot{p}$ the projection $\dot{E} \to X$. For example, we use the notations $\dot{\pi}_X\colon \dot{T}^*X \to X$, $\dot{T}^*_ZX$, etc.

Definition 1.2.2. *A kernel data is a triple (X, Z, σ), where X is a manifold, Z is a closed submanifold of X and σ is a section of $T^*X \times_X Z \to Z$.*

We set $\mathcal{T}(\sigma) = \sigma^{-1}(T^*_ZX)$ and $\mathcal{Z}(\sigma) = \sigma^{-1}(T^*_XX)$. We have therefore

$$\mathcal{Z}(\sigma) \subset \mathcal{T}(\sigma) \subset Z.$$

Each kernel data (X, Z, σ) defines a closed cone P_σ in $T_ZX \times_X \mathcal{T}(\sigma)$ by

$$P_\sigma = \{(x, v) \in T_ZX; x \in \mathcal{T}(\sigma) \text{ and } \langle v, \sigma(x)\rangle \geqslant 0\}.$$

Consider the deformation of the normal bundle to Z in X which will be denoted by $\widetilde{X}_Z$ or simply by $\widetilde{X}$ (see e.g. [KS2]). We have the following commutative diagram where the squares marked by $\square$ are cartesian:

$$\begin{tikzcd}
 & \{0\} \arrow[r, hook] & \mathbb{R} & \{t \in \mathbb{R}; t > 0\} \arrow[l, hook'] \\
P_\sigma \arrow[r, hook] & T_ZX \arrow[u] \arrow[r, hook, "s"] \arrow[d, "\tau_Z"] \arrow[ur, phantom, "\square"] & \widetilde{X}_Z \arrow[u, "t"] \arrow[d, "p"] \arrow[ur, phantom, "\square"] & \Omega \arrow[u] \arrow[l, hook', "j"'] \arrow[dl, "\widetilde{p}"] \\
 & Z \arrow[r, hook, "i"] & X. &
\end{tikzcd} \tag{1.6}$$

Here Ω is the open subset defined by $\Omega = \{t > 0\}$ for the natural smooth map $t\colon \widetilde{X}_Z \to \mathbb{R}$. The normal bundle T_ZX is identified with the inverse image of $0 \in \mathbb{R}$ by t. With a local coordinate system $(x, z) = (x_1, \dots, x_n, z_1, \dots, z_m)$ of X such that Z is given by $x = 0$, $\widetilde{X}_Z$ has the coordinates $(t, \tilde{x}, z) = (t, \tilde{x}_1, \dots, \tilde{x}_n, z_1, \dots, z_m)$ and p is given by $p(t, \tilde{x}, z) = (t\tilde{x}, z)$.

Recall that the normal cone $C_Z(A)$ of a subset A of X is a closed cone of T_ZX defined by

$$C_Z(A) = T_ZX \cap \overline{p^{-1}(A) \cap \Omega}. \tag{1.7}$$

Note that p is not smooth but the relative dualizing complex $\omega_{\widetilde{X}/X}$ is isomorphic to $\mathbb{K}_{\widetilde{X}}[1]$. In the sequel we will usually regard P_σ as a closed subset of $\widetilde{X}_Z$ by $P_\sigma \subset T_ZX \subset \widetilde{X}_Z$.

Definition 1.2.3. *(i) Let (X, Z, σ) be a kernel data. We define the kernel $\mathcal{L}_\sigma(Z, X) \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$ by*

$$\mathcal{L}_\sigma(Z, X) = \mathrm{R}p_{!!}(\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_{P_\sigma}) \otimes \beta_X(\mathrm{R}i_*\omega_{Z/X}^{\otimes -1}).$$

(ii) A morphism of kernel data $f\colon (X_1, Z_1, \sigma_1) \to (X_2, Z_2, \sigma_2)$ is a morphism of manifolds $f\colon X_1 \to X_2$ satisfying

(i) $f(Z_1) \subset Z_2$,
(ii) $\sigma_1 = f^*\sigma_2$.

Remark 1.2.4. Note that $\mathcal{L}_\sigma(Z,X)$ is supported on $\mathcal{T}(\sigma)$, *i.e.*

$$\mathcal{L}_\sigma(Z,X) \xrightarrow{\sim} \mathcal{L}_\sigma(Z,X) \otimes \widetilde{\mathbb{K}}_{\mathcal{T}(\sigma)} .$$

This kernel behaves differently on $\mathcal{Z}(\sigma)$ and outside. We have

$$\mathcal{L}_\sigma(Z,X) \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)} \simeq \mathbb{K}_Z \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)}$$

and $\mathcal{L}_\sigma(Z,X)|_{X\setminus\mathcal{Z}(\sigma)}$ is concentrated in degree $-\operatorname{codim} Z$ (see Corollary 1.2.13).

In order to prove these facts, we shall start by the following vanishing lemma.

Lemma 1.2.5. (i) $\mathrm{R}p_{!!}(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{T_ZX}) \simeq 0$ *and* $\mathrm{R}p_{!!}(\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_{T_ZX}) \simeq \mathrm{R}i_*\omega_{Z/X}$.
(ii) *Regarding Z as the zero section of $T_ZX \subset \widetilde{X}_Z$, we have*

$$\mathrm{R}p_{!!}\left(\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_Z\right) \simeq \widetilde{\mathbb{K}}_Z .$$

(iii) $\left(\mathrm{R}p_{!!}(\mathbb{K}_{T_ZX} \otimes \widetilde{\mathbb{K}}_{P_\sigma})\right) \otimes \widetilde{\mathbb{K}}_{Z\setminus\mathcal{Z}(\sigma)} \simeq 0$.

Proof. (i) Since the problem is local, we may assume that X is affine endowed with a system of global coordinates (x,z) such that $Z=\{x=0\}$, $\widetilde{X}_Z = (t,\widetilde{x},z)$ and $p(t,\widetilde{x},z) = (t\widetilde{x},z)$. We have then for all integer j

$$\mathrm{R}^jp_{!!}\big(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{T_ZX}\big) \simeq \mathrm{R}^jp_{!!}\Big(\underset{R>0,\,\varepsilon>0}{\text{“}\varinjlim\text{”}}\ \mathbb{K}_{\{0<t\leqslant\varepsilon,\,|\widetilde{x}|<R\}}\Big) \simeq \underset{R>0,\,\varepsilon>0}{\text{“}\varinjlim\text{”}}\ \mathrm{R}^jp_!\,\mathbb{K}_{\{0<t\leqslant\varepsilon,\,|\widetilde{x}|<R\}} \simeq 0,$$

which implies the first statement. The last one follows from the distinguished triangle

$$\mathrm{R}p_{!!}(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{T_ZX}) \longrightarrow \mathrm{R}p_{!!}(\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_{T_ZX}) \longrightarrow \mathrm{R}p_{!!}(\mathbb{K}_{T_ZX}) \xrightarrow{+1}$$

and $\mathrm{R}p_{!!}(\mathbb{K}_{T_ZX}) \simeq \mathrm{R}i_*\omega_{Z/X}$.

(ii) We have a chain of morphisms

$$\mathrm{R}p_{!!}\left(\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_Z\right) \to \mathrm{R}p_{!!}\left(\mathbb{K}_{\overline{\Omega}} \otimes \mathbb{K}_Z\right) \simeq \mathrm{R}p_{!!}\,\mathbb{K}_Z \simeq \mathbb{K}_Z .$$

which allows us to prove the isomorphism locally on X. With the coordinate system as above, we get for all integer j

$$\begin{aligned}
\mathrm{R}^jp_{!!}\left(\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_Z\right) &\simeq \mathrm{R}^jp_{!!}\Big(\underset{\varepsilon>0}{\text{“}\varinjlim\text{”}}\,\mathbb{K}_{\{0\leqslant t\leqslant\varepsilon,|\widetilde{x}|\leqslant\varepsilon\}}\Big) \simeq \underset{\varepsilon>0}{\text{“}\varinjlim\text{”}}\,\mathrm{R}^jp_!\,\mathbb{K}_{\{0\leqslant t\leqslant\varepsilon,|\widetilde{x}|\leqslant\varepsilon\}} \\
&\simeq \begin{cases} \underset{\varepsilon>0}{\text{“}\varinjlim\text{”}}\,\mathbb{K}_{\{|x|\leqslant\varepsilon^2\}} \simeq \widetilde{\mathbb{K}}_Z & \text{if } j=0, \\ 0 & \text{if } j\neq 0. \end{cases}
\end{aligned}$$

(iii) For $z_0 \in \mathcal{T}(\sigma)\setminus\mathcal{Z}(\sigma)$, we have

$$\Big(\mathrm{R}p_{!!}(\mathbb{K}_{T_ZX} \otimes \widetilde{\mathbb{K}}_{P_\sigma})\Big) \otimes \widetilde{\mathbb{K}}_{z_0} \simeq \mathrm{R}p_{!!}\left(\mathbb{K}_{T_ZX} \otimes \widetilde{\mathbb{K}}_{P_\sigma\cap p^{-1}(z_0)}\right).$$

Set $\sigma(z_0) = \langle\xi_0, dx\rangle \neq 0$. Then we have

$$\mathbb{K}_{T_ZX} \otimes \widetilde{\mathbb{K}}_{P_\sigma\cap p^{-1}(z_0)} \simeq \underset{R>0,\varepsilon>0}{\text{“}\varinjlim\text{”}}\ \mathbb{K}_{\{t=0,\ -\varepsilon\leq\langle\xi_0,\tilde{x}\rangle,\,|\tilde{x}|<R\}},$$

and for all integer j

$$\big(\mathrm{R}^j p_{!!}(\mathbb{K}_{T_Z X} \otimes \widetilde{\mathbb{K}}_{P_\sigma})\big) \otimes \widetilde{\mathbb{K}}_{z_0} \simeq \widetilde{\mathbb{K}}_{z_0} \otimes \text{“}\varinjlim_{R>0,\,\varepsilon>0}\text{”}\ \mathrm{R}^j p_!\big(\mathbb{K}_{\{t=0,\ -\varepsilon \le \langle \xi_0, \tilde{x}\rangle,\, |\tilde{x}|<R\}}\big) \simeq 0.$$

□

Lemma 1.2.6. *There is a natural morphism*

$$\mathcal{L}_\sigma(Z, X) \longrightarrow \widetilde{\mathbb{K}}_{\mathcal{T}(\sigma)} \otimes \beta_X \left(\mathrm{R}i_* \omega_{Z/X}^{\otimes -1}\right).$$

Proof. Regard $\mathcal{T}(\sigma)$ as a subset of $\widetilde{X}_Z$ by $\mathcal{T}(\sigma) \subset Z \subset T_Z X \subset \widetilde{X}_Z$. Then we get a natural morphism

$$\mathcal{L}_\sigma(Z, X) \to \mathrm{R}p_{!!}\left(\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_{\mathcal{T}(\sigma)}\right) \otimes \beta_X \left(\mathrm{R}i_* \omega_{Z/X}^{\otimes -1}\right).$$

Hence the desired morphism is obtained by Lemma 1.2.5 (ii). □

The following lemma provides a useful distinguished triangle to study some properties of the kernel $\mathcal{L}_\sigma(Z, X)$.

Lemma 1.2.7. *There is a natural distinguished triangle*

$$\mathrm{R}p_{!!}\big(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\big) \otimes \beta_X\big(\mathrm{R}i_* \omega_{Z/X}^{\otimes -1}\big) \longrightarrow \mathcal{L}_\sigma(Z, X) \longrightarrow \mathrm{R}p_{!!}\big(\mathbb{K}_{T_Z X} \otimes \widetilde{\mathbb{K}}_{P_\sigma}\big) \otimes \mathrm{R}i_* \omega_{Z/X}^{\otimes -1} \xrightarrow{+1}.$$

Proof. It is enough to apply the triangulated functor $\mathrm{R}p_{!!}\big(\,\cdot\, \otimes \widetilde{\mathbb{K}}_{P_\sigma}\big) \otimes \beta_X(\mathrm{R}i_* \omega_{Z/X}^{\otimes -1})$ to the distinguished triangle

$$\mathbb{K}_\Omega \longrightarrow \mathbb{K}_{\overline{\Omega}} \longrightarrow \mathbb{K}_{T_Z^* X} \xrightarrow{+1}, \tag{1.8}$$

and to use $\mathbb{K}_Z \otimes \beta_X(\mathrm{R}i_* \omega_{Z/X}^{\otimes -1}) \simeq \mathrm{R}i_* \omega_{Z/X}^{\otimes -1}$. □

Recall that $\mathcal{Z}(\sigma)$ is the set of zeroes of σ, i.e. $\mathcal{Z}(\sigma) = \sigma^{-1}(T_X^* X) \subset Z$.

Proposition 1.2.8. *We have*

$$\mathcal{L}_\sigma(Z, X) \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)} \simeq \mathbb{K}_Z \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)}.$$

In particular, if $\sigma = 0$, then $\mathcal{L}_\sigma(Z, X) \simeq \mathbb{K}_Z$.

Proof. By the definition of $\mathcal{Z}(\sigma)$, the cone $P_\sigma \underset{Z}{\times} \mathcal{Z}(\sigma)$ coincides with $T_Z X \underset{Z}{\times} \mathcal{Z}(\sigma)$. Hence we have $\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_{P\sigma} \otimes p^{-1} \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)} \simeq \mathbb{K}_{\overline{\Omega}} \otimes p^{-1} \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)}$, which implies

$$\mathcal{L}_\sigma(Z, X) \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)} \simeq \mathrm{R}p_{!!}(\mathbb{K}_{\overline{\Omega}} \otimes \widetilde{\mathbb{K}}_{T_Z X}) \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)} \otimes \beta_X(\mathrm{R}i_* \omega_{Z/X}^{\otimes -1}).$$

Hence the result follows from Lemma 1.2.5 (i). □

Proposition 1.2.9. *Let (X, Z, σ) be a kernel data, and set $X_0 = X \setminus \mathcal{Z}(\sigma)$ and $Z_0 = Z \setminus \mathcal{Z}(\sigma)$. Then there is a natural distinguished triangle*

$$\mathrm{R}j_{!!}\mathcal{L}_{\sigma_0}(Z_0, X_0) \longrightarrow \mathcal{L}_\sigma(Z, X) \longrightarrow \mathbb{K}_Z \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)} \xrightarrow{+1},$$

where σ_0 is the restriction of σ to Z_0 and j denotes the open immersion $X_0 \hookrightarrow X$.

Proof. We have the distinguished triangle

$$\mathcal{L}_\sigma(Z, X) \otimes \widetilde{\mathbb{K}}_{X_0} \longrightarrow \mathcal{L}_\sigma(Z, X) \longrightarrow \mathcal{L}_\sigma(Z, X) \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)} \xrightarrow{+1}.$$

The first term is isomorphic to $\mathrm{R}j_{!!}\mathcal{L}_{\sigma_0}(Z_0, X_0)$, and the last term is isomorphic to $\mathbb{K}_Z \otimes \widetilde{\mathbb{K}}_{\mathcal{Z}(\sigma)}$ by Lemma 1.2.8. □

Corollary 1.2.10. *There are natural morphisms*

$$\mathbb{K}_Z \longrightarrow \mathcal{L}_\sigma(Z,X) \longrightarrow \widetilde{\mathbb{K}}_{\mathcal{T}(\sigma)} \otimes \beta_X\left(\mathrm{R}i_*\omega_{Z/X}^{\otimes -1}\right).$$

Proof. The first arrow is constructed as an immediate consequence of the preceding proposition and the obvious inclusion $P_\sigma \subset P_0 = T_Z X$. The last arrow follows from Lemma 1.2.6. □

Proposition 1.2.11. *Assume the section σ never vanishes. Then*

$$\mathcal{L}_\sigma(Z,X) \simeq \mathrm{R}p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\right) \otimes \beta_X\left(\mathrm{R}i_*\omega_{Z/X}^{\otimes -1}\right) \simeq \text{“}\varinjlim_{U}\text{”}\, \mathbb{K}_U \otimes \beta_X\left(\mathrm{R}i_*\omega_{Z/X}^{\otimes -1}\right) \otimes \widetilde{\mathbb{K}}_{\mathcal{T}(\sigma)},$$

where the inductive limit is taken over the family of open subsets U of X such that

$$P_\sigma \cap \mathrm{C}_Z(U) \subset Z.$$

Here, Z is regarded as the zero section of $T_Z X$.

Remark that the set of such U's is a filtrant ordered set by the inclusion order.

Proof. By Lemma 1.2.7 and Lemma 1.2.5 (iii), we have

$$\mathcal{L}_\sigma(Z,X) \simeq \mathrm{R}p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\right) \otimes \beta_X\left(\mathrm{R}i_*\omega_{Z/X}^{\otimes -1}\right).$$

Hence it is enough to show

$$\mathrm{R}p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\right) \simeq \text{“}\varinjlim_{U}\text{”}\, \mathbb{K}_U \otimes \widetilde{\mathbb{K}}_{\mathcal{T}(\sigma)}.$$

Since we have $Z \cap U = \varnothing$ on a neighborhood of $\mathcal{T}(\sigma)$, $p^{-1}(U) \cap \Omega = p^{-1}(U) \cap \overline{\Omega}$ is a closed subset of Ω and we get the following chain of natural morphisms :

$$p^{-1}\mathbb{K}_U \simeq \mathbb{K}_{p^{-1}(U)} \longrightarrow \mathbb{K}_{p^{-1}(U)\cap\Omega} \longrightarrow \mathbb{K}_\Omega \longrightarrow \mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}.$$

Since $\overline{p^{-1}(U) \cap \Omega} \cap P_\sigma = \mathrm{C}_Z(U) \cap P_\sigma$ is contained in the zero section of $T_Z X$, $\mathrm{Supp}(p^{-1}\mathbb{K}_U \otimes \widetilde{\mathbb{K}}_{P_\sigma})$ is proper over Z. Hence we have a chain of morphisms

$$\mathbb{K}_U \longrightarrow p_*(p^{-1}\mathbb{K}_U \otimes \widetilde{\mathbb{K}}_{P_\sigma}) \simeq p_{!!}(p^{-1}\mathbb{K}_U \otimes \widetilde{\mathbb{K}}_{P_\sigma}) \longrightarrow p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\right),$$

which provides a natural morphism

$$\text{“}\varinjlim_{U}\text{”}\, \mathbb{K}_U \longrightarrow \mathrm{R}p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\right).$$

By tensorisation we get the morphism

$$\text{“}\varinjlim_{U}\text{”}\, \mathbb{K}_U \otimes \widetilde{\mathbb{K}}_{\mathcal{T}(\sigma)} \longrightarrow \mathrm{R}p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\right). \tag{1.9}$$

We shall now show that this morphism is an isomorphism. It is enough to show that (1.9) is an isomorphism after tensoring by $\widetilde{\mathbb{K}}_{x_0}$ for any $x_0 \in \mathcal{T}(\sigma)$. Let us take local coordinate system (x,z) of X such that $Z = \{x = 0\}$. We may assume $x_0 = (0,0)$, and we set $\sigma(x_0) = \langle \xi_0, dx\rangle$. We then have

$$\mathrm{R}p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\right) \otimes \widetilde{\mathbb{K}}_{x_0} \simeq \mathrm{R}p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma} \otimes \widetilde{\mathbb{K}}_{p^{-1}(x_0)}\right) \simeq \mathrm{R}p_{!!}\left(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma \cap p^{-1}(x_0)}\right),$$

and

$$\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma \cap p^{-1}(x_0)} \simeq \text{“}\varinjlim_{V \subset\subset \widetilde{X}_Z,\ P_\sigma \cap p^{-1}(x_0) \subset V'}\text{”}\, \mathbb{K}_{\Omega \cap V \cap \overline{V'}} \simeq \widetilde{\mathbb{K}}_{x_0} \otimes \text{“}\varinjlim_{R>0,\,\varepsilon_1>0,\,\varepsilon_2>0}\text{”}\, \mathbb{K}_{A_{R,\varepsilon_1,\varepsilon_2}},$$

where we have set

$$A_{R,\varepsilon_1,\varepsilon_2} = \Big\{(t,\widetilde{x},z) \in \widetilde{X}_Z\ ;\ 0 < t \leqslant \varepsilon_1,\ -\varepsilon_2 \leqslant \langle \xi_0,\, \widetilde{x}\rangle, |\widetilde{x}| < R\Big\}.$$

Hence for all integer j, we have

$$\mathrm{R}^j p_{!!}\Big(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\Big) \otimes \widetilde{\mathbb{K}}_{x_0} \simeq \widetilde{\mathbb{K}}_{x_0} \otimes \underset{R>0,\,\varepsilon_1>0,\,\varepsilon_2>0}{\text{“}\varinjlim\text{”}} \mathrm{R}^j p_! \,\mathbb{K}_{A_{R,\varepsilon_1,\varepsilon_2}}\,.$$

We have

$$\begin{aligned} p^{-1}((x,z)) &\simeq \{t \in \mathbb{R}; 0 < t \leq \varepsilon_1,\ -\varepsilon_2 \leq \langle \xi_0, t^{-1}x\rangle,\ |t^{-1}x| < R\} \\ &\simeq \{t \in \mathbb{R}; R^{-1}|x| < t \leq \varepsilon_1,\ -\varepsilon_2^{-1}\langle \xi_0, x\rangle \leq t\}, \end{aligned}$$

and hence

$$\mathrm{R}p_!(\mathbb{K}_{A_{R,\varepsilon_1,\varepsilon_2}}) \simeq \mathbb{K}_{\left\{R^{-1}|x|<-\varepsilon_2^{-1}\langle x,\,\xi_0\rangle\leqslant\varepsilon_1\right\}}\,.$$

Taking the limit we can use a cofinality argument to get

$$\mathrm{R}p_{!!}\Big(\mathbb{K}_\Omega \otimes \widetilde{\mathbb{K}}_{P_\sigma}\Big) \otimes \widetilde{\mathbb{K}}_{x_0} \simeq \widetilde{\mathbb{K}}_{x_0} \otimes \underset{\varepsilon>0}{\text{“}\varinjlim\text{”}}\, \mathbb{K}_{\{(x,z)\in X\ ;\ -\langle\xi_0,x\rangle>\varepsilon|x|\}}\,.$$

Then the theorem follows from the following easy sublemma. ☐

Sublemma 1.2.12. (i) *Let $U = \{(x,z) \in X; \varepsilon|x| < -\langle \xi_0, x\rangle\}$. Then $P_\sigma \cap \mathrm{C}_Z(U) \subset Z$.*

(ii) *Let $U \subset X$ be an open subset such that $P_\sigma \cap \mathrm{C}_Z(U) \subset Z$. Then there exist $\varepsilon > 0$ and $\delta > 0$ such that*

$$U \cap \{|(x,z)| \leqslant \delta\} \subset \{(x,z) \in X; -\langle x, \xi_0\rangle > \varepsilon|x|\}.$$

Corollary 1.2.13. *Let (X,Z,σ) be a kernel data. Assume that X is endowed with a local coordinate system (x,z) such that $Z = \{x = 0\}$ and σ is a nowhere vanishing section. Then, writing $\sigma(z) = \langle \sigma_1(z), dx\rangle + \langle \sigma_2(z), dz\rangle$, we have*

$$\mathcal{L}_\sigma\,(Z,X) \simeq \widetilde{\mathbb{K}}_{\{x=0,\,\sigma_2(z)=0\}} \otimes \underset{\varepsilon>0}{\text{“}\varinjlim\text{”}}\, \mathbb{K}_{\left\{(x,z);-\langle\sigma_1(z),x\rangle>\varepsilon|x|\right\}}\,[\operatorname{codim} Z].$$

Remark 1.2.14. (i) We have

$$\alpha_X\big(\mathcal{L}_\sigma(Z,X)\big) \simeq \mathbb{K}_{\mathcal{Z}(\sigma)}\,.$$

(ii) Let (X,Z,σ_1) and (X,Z,σ_2) be kernel data, and let W be a closed subset of Z such that $\sigma_1(x) = \sigma_2(x)$ for all $x \in W$. Since $P_{\sigma_1} \cap \tau_Z^{-1}W = P_{\sigma_2} \cap \tau_Z^{-1}W$, we have

$$\mathcal{L}_{\sigma_1}(X,Z) \otimes \widetilde{\mathbb{K}}_W \simeq \mathcal{L}_{\sigma_2}(X,Z) \otimes \widetilde{\mathbb{K}}_W\,.$$

1.3. **Functorial Properties.** In this subsection, we will investigate the behavior of microlocal kernels $\mathcal{L}_\sigma(Z,X)$ under inverse and proper direct images, and under convolution.

Let $f\colon (X_1,Z_1,\sigma_1) \to (X_2,Z_2,\sigma_2)$ be morphism of kernel data. We have the diagrams of manifolds

$$\begin{tikzcd} T^*_{Z_1}X_1 & T^*_{Z_2}X_2 \underset{Z_2}{\times} Z_1 \arrow[l, "f_d"'] \arrow[r, "f_\pi"] & T^*_{Z_2}X_2 \\ \mathcal{T}(\sigma_1) \arrow[u, "\sigma_1"] & \mathcal{T}(\sigma_2) \underset{Z_2}{\times} Z_1 \arrow[l, hook'] \arrow[u] \arrow[r] & \mathcal{T}(\sigma_2) \arrow[u, "\sigma_2"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{X}_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & \widetilde{X}_2 \arrow[r, "t"] \arrow[d, "p_2"] & \mathbb{R} \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

where $\widetilde{X}_k = \widetilde{X_k}_{Z_k}$ $(k = 1, 2)$. We denote by $i_k \colon Z_k \hookrightarrow X_k$ the inclusion map. We have

$$P_{\sigma_1} \underset{X_2}{\times} \mathcal{T}(\sigma_2) = \tilde{f}^{-1}(P_{\sigma_2}). \tag{1.10}$$

Proposition 1.3.1. *Let $f \colon (X_1, Z_1, \sigma_1) \to (X_2, Z_2, \sigma_2)$ be a morphism of kernel data. Assume that $Z_1 = f^{-1}(Z_2)$ and the morphism $f \colon X_1 \to X_2$ is clean with respect to Z_2 (i.e. $(T_{Z_1}X_1)_x \to (T_{Z_2}X_2)_{f(x)}$ is injective for any $x \in Z_1$). Then there exists a natural morphism*

$$f^{-1}\mathcal{L}_{\sigma_2}(Z_2, X_2) \to \mathcal{L}_{\sigma_1}(Z_1, X_1) \otimes \beta_{X_1}(\mathrm{R}i_{1*}\omega_{Z_1/Z_2}) \otimes \omega^{\otimes -1}_{X_1/X_2} \otimes \widetilde{\mathbb{K}}_{f^{-1}\mathcal{T}(\sigma_2)}\,.$$

Proof. Since f is clean, $\widetilde{X}_1 \to \widetilde{X}_2 \underset{X_2}{\times} X_1$ is a closed embedding and there is a morphism of functors $f^{-1}\,\mathrm{R}p_{2!!} \to \mathrm{R}p_{1!!}\tilde{f}^{-1}$ which induces a natural morphism

$$\begin{aligned} f^{-1}\mathcal{L}_{\sigma_2}(Z_2, X_2) &\simeq f^{-1}\,\mathrm{R}p_{2!!}\left(\mathbb{K}_{\overline{\Omega}_2} \otimes \widetilde{\mathbb{K}}_{P_{\sigma_2}}\right) \otimes f^{-1}\beta_{X_2}(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}) \\ &\to \mathrm{R}p_{1!!}\tilde{f}^{-1}\left(\mathbb{K}_{\overline{\Omega}_2} \otimes \widetilde{\mathbb{K}}_{P_{\sigma_2}}\right) \otimes f^{-1}\beta_{X_2}(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}) \\ &\simeq \mathrm{R}p_{1!!}\left(\mathbb{K}_{\overline{\Omega}_1} \otimes \widetilde{\mathbb{K}}_{\tilde{f}^{-1}(P_{\sigma_2})}\right) \otimes \beta_{X_1}(f^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}). \end{aligned} \tag{1.11}$$

By (1.10), we have a morphism

$$f^{-1}\mathcal{L}_{\sigma_2}(Z_2, X_2) \to \mathrm{R}p_{1!!}\left(\mathbb{K}_{\overline{\Omega}_1} \otimes \widetilde{\mathbb{K}}_{P_{\sigma_1}}\right) \otimes f^{-1}(\widetilde{\mathbb{K}}_{\mathcal{T}(\sigma_2)}) \otimes \beta_{X_1}(f^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}). \tag{1.12}$$

Hence, to get the desired morphism, it is enough to remark that

$$f^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2} \simeq \mathrm{R}i_{1*}\left(\omega^{\otimes -1}_{Z_1/X_1} \otimes \omega_{Z_1/Z_2} \otimes i_1^{-1}\omega^{\otimes -1}_{X_1/X_2}\right) \simeq \mathrm{R}i_{1*}\omega^{\otimes -1}_{Z_1/X_1} \otimes \mathrm{R}i_{1*}\omega_{Z_1/Z_2} \otimes \omega^{\otimes -1}_{X_1/X_2}.$$

□

By adjunction, we obtain:

Corollary 1.3.2. *Under the hypothesis of the Proposition 1.3.1, we have a natural morphism*

$$\mathcal{L}_{\sigma_2}(Z_2, X_2) \to \mathrm{R}f_*\left(\mathcal{L}_{\sigma_1}(Z_1, X_1) \otimes \beta_{X_1}(\mathrm{R}i_{1*}\omega_{Z_1/Z_2}) \otimes \omega^{\otimes -1}_{X_1/X_2} \otimes f^{-1}\,\widetilde{\mathbb{K}}_{\mathcal{T}(\sigma_2)}\right).$$

Proposition 1.3.3. *Let $f \colon (X_1, Z_1, \sigma_1) \to (X_2, Z_2, \sigma_2)$ be a morphism of kernel data. Assume that $f^{-1}(Z_2) = Z_1$ and f is transversal to Z_2. Then we have a natural isomorphism*

$$f^{-1}\mathcal{L}_{\sigma_2}(Z_2, X_2) \xrightarrow{\sim} \mathcal{L}_{\sigma_1}(Z_1, X_1).$$

Proof. Indeed if f is transversal, $\widetilde{X}_1 \to \widetilde{X}_2 \underset{X_2}{\times} X_1$ is an isomorphism and $Z_1 \cap f^{-1}(\mathcal{T}(\sigma_2)) = \mathcal{T}(\sigma_1)$, which implies that the morphism (1.11) as well as (1.12) is an isomorphism. We have furthermore $\omega_{Z_1/Z_2} \simeq i_1^{-1}\omega_{X_1/X_2}$. □

Proposition 1.3.4. *Let $f \colon (X_1, Z_1, \sigma_1) \to (X_2, Z_2, \sigma_2)$ be a morphism of kernel data. Then there is a natural morphism*

$$\mathrm{R}f_{!!}\Big(\mathcal{L}_{\sigma_1}(Z_1, X_1) \otimes \beta_{X_1}(\mathrm{R}i_{1*}\omega_{Z_1/Z_2})\Big) \to \mathcal{L}_{\sigma_2}(Z_2, X_2).$$

Proof. The left hand side is isomorphic to

$$\begin{aligned}
&\mathrm{R}f_{!!}\Big(\mathrm{R}p_{1!!}(\mathbb{K}_{\overline{\Omega}_1}\otimes\widetilde{\mathbb{K}}_{P_{\sigma_1}})\otimes\beta_{X_1}(\mathrm{R}i_{1*}\omega^{\otimes -1}_{Z_1/X_1})\otimes\beta_{X_1}(\mathrm{R}i_{1*}\omega_{Z_1/Z_2})\Big)\\
&\qquad\simeq \mathrm{R}f_{!!}\Big(\mathrm{R}p_{1!!}(\mathbb{K}_{\overline{\Omega}_1}\otimes\widetilde{\mathbb{K}}_{P_{\sigma_1}})\otimes\omega_{X_1/X_2}\otimes\beta_{X_1}(f^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2})\Big)\\
&\qquad\simeq \mathrm{R}f_{!!}\,\mathrm{R}p_{1!!}\Big((\mathbb{K}_{\overline{\Omega}_1}\otimes\widetilde{\mathbb{K}}_{P_{\sigma_1}})\otimes p_1^{-1}\omega_{X_1/X_2}\Big)\otimes\beta_{X_2}(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2})\\
&\qquad\simeq \mathrm{R}p_{2!!}\,\mathrm{R}\widetilde{f}_{!!}\Big(\widetilde{f}^{-1}\,\mathbb{K}_{\overline{\Omega}_2}\otimes\widetilde{\mathbb{K}}_{P_{\sigma_1}}\otimes p_1^{-1}\omega_{X_1/X_2}\Big)\otimes\beta_{X_2}(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2})\\
&\qquad\simeq \mathrm{R}p_{2!!}\Big(\mathbb{K}_{\overline{\Omega}_2}\otimes\mathrm{R}\widetilde{f}_{!!}(\widetilde{\mathbb{K}}_{P_{\sigma_1}}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2})\Big)\otimes\beta_{X_2}(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}).
\end{aligned}\tag{1.13}$$

Hence, it is enough to construct a morphism

$$\mathrm{R}\widetilde{f}_{!!}\left(\widetilde{\mathbb{K}}_{P_{\sigma_1}}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2}\right)\longrightarrow\widetilde{\mathbb{K}}_{P_{\sigma_2}}\,.\tag{1.14}$$

By adjunction it is enough to construct a morphism $\widetilde{\mathbb{K}}_{P_{\sigma_1}}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2}\longrightarrow\widetilde{f}^{!}\,\widetilde{\mathbb{K}}_{P_{\sigma_2}}$. However by (1.10), we have

$$\widetilde{\mathbb{K}}_{P_{\sigma_1}}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2}\longrightarrow\widetilde{\mathbb{K}}_{P_{\sigma_1}\times_{X_2}\mathcal{T}(\sigma_2)}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2}\simeq\widetilde{f}^{-1}\,\widetilde{\mathbb{K}}_{P_{\sigma_2}}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2}\simeq\widetilde{f}^{!}\,\widetilde{\mathbb{K}}_{P_{\sigma_2}},$$

where the last isomorphism follows from (1.3 a). ☐

Corollary 1.3.5. *Let $f\colon(X_1,Z_1,\sigma_1)\to(X_2,Z_2,\sigma_2)$ be a morphism, and assume that f is smooth and induces an isomorphism from Z_1 to Z_2. Then we have a natural isomorphism*

$$\mathrm{R}f_{!!}\mathcal{L}_{\sigma_1}(Z_1,X_1)\xrightarrow{\ \sim\ }\mathcal{L}_{\sigma_2}(Z_2,X_2).$$

Proof. By the assumption, we have $\mathcal{T}(\sigma_2)\times_{Z_2}Z_1=\mathcal{T}(\sigma_1)$. By (1.13), it is enough to prove that (1.14) is an isomorphism. Since $P_{\sigma_1}=\widetilde{f}^{-1}(P_{\sigma_2})$, we have

$$\mathrm{R}\widetilde{f}_{!!}\left(\widetilde{\mathbb{K}}_{P_{\sigma_1}}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2}\right)\ \simeq\ \widetilde{\mathbb{K}}_{P_{\sigma_2}}\otimes\mathrm{R}\widetilde{f}_{!!}\left(\widetilde{\mathbb{K}}_{T_{Z_1}X_1}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2}\right).$$

Hence we have reduced the problem to

$$\mathrm{R}\widetilde{f}_{!!}\left(\widetilde{\mathbb{K}}_{T_{Z_1}X_1}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2}\right)\simeq\widetilde{\mathbb{K}}_{T_{Z_2}X_2}\,.$$

Since f is smooth, we can take local coordinate systems (x,z) on X_2 and (x,y,z) on X_1 such that $Z_2=\{x=0\}$, $Z_1=\{x=0,\,y=0\}$ and f is given by the projection. We then take a coordinate system $(t,\tilde{x},z)$ on $\widetilde{X_2}$ and $(t,\tilde{x},\tilde{y},z)$ on $\widetilde{X_1}$. The associated morphism $\widetilde{f}\colon\widetilde{X_1}\to\widetilde{X_2}$ is given by $(t,\tilde{x},\tilde{y},z)\to(t,\tilde{x},z)$. Then we can check easily $\mathrm{R}\widetilde{f}_{!!}(\widetilde{\mathbb{K}}_{T_{Z_1}X_1}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2})\simeq\mathrm{R}\widetilde{f}_{!!}(\widetilde{\mathbb{K}}_{\{t=0\}}\otimes\omega_{\widetilde{X}_1/\widetilde{X}_2})\simeq\widetilde{\mathbb{K}}_{\{t=0\}}$. ☐

Lemma 1.3.6. *Let (X,Z,σ) be a kernel data on X, and let $f\colon X\to Y$ be a smooth morphism which induces a closed embedding $Z\hookrightarrow Y$. Assume that $\sigma(x)\notin T^*_{f(x)}Y$ for any $x\in\mathcal{T}(\sigma)$. Then we have*

$$\mathrm{R}f_{!!}\mathcal{L}_\sigma(Z,X)\simeq 0.$$

Proof. For any $x_0\in\mathcal{T}(\sigma)$, take a local coordinate system $(y,z)=(y_1,\dots,y_n,z_1,\dots,z_m)$ of Y in a neighborhood of $f(x_0)$ such that $f(Z)$ is given by $y=0$. Then we can take a local coordinate system (t,x,y,z) of X in a neighborhood of x_0 such that Z is given by $\{t=0,\,x=0,\,y=0\}$, and $\sigma(x_0)=-dt(x_0)$. Then we have

$$\mathcal{L}_\sigma(Z,X)\otimes\widetilde{\mathbb{K}}_{x_0}\simeq\Big(\underset{\delta>0,\,\varepsilon>0}{\text{“}\varinjlim\text{”}}\ \mathbb{K}_{F_{\delta,\varepsilon}}\Big)\otimes\beta_X(\mathrm{R}i_*\omega^{\otimes -1}_{Z/X})\otimes\widetilde{\mathbb{K}}_{x_0},$$

where
$$F_{\delta,\varepsilon} = \{(t,x,y,z); \delta \geq t > \varepsilon(|x|+|y|)\}.$$
Hence, $\big(\mathrm{R}f_{!!}(\mathcal{L}_\sigma(Z,X)\big) \otimes \widetilde{\mathbb{K}}_{f(x_0)} \simeq \mathrm{R}f_{!!}(\mathcal{L}_\sigma(Z,X) \otimes \widetilde{\mathbb{K}}_{x_0}) \simeq 0$ follows from
$$\mathrm{R}^j f_!(\mathbb{K}_{F_{\delta,\varepsilon}}) \simeq 0 \quad \text{for any } j \in \mathbb{Z}.$$
□

Proposition 1.3.7. *Let $f\colon (X_1, Z_1, \sigma_1) \to (X_2, Z_2, \sigma_2)$ be a morphism of kernel data, and assume that f is a closed immersion which induces an isomorphism $Z_1 \xrightarrow{\sim} Z_2$. Then there is a natural isomorphism*
$$\mathcal{L}_{\sigma_1}(Z_1, X_1) \xrightarrow{\sim} f^!\mathcal{L}_{\sigma_2}(Z_2, X_2).$$

Proof. Since f is a closed immersion, we get the commutative diagrams
$$\begin{tikzcd}
Z_1 \arrow[r, hook, "i_1"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\square"] & X_1 \arrow[d, hook, "f"] & \widetilde{X}_1 \arrow[l, "p_1"'] \arrow[d, hook, "\widetilde{f}"] \arrow[dr, phantom, "\square"] & \Omega_1 \arrow[l, hook', "j_1"'] \arrow[d, hook] \\
Z_2 \arrow[r, hook, "i_2"'] & X_2 & \widetilde{X}_2 \arrow[l, "p_2"] & \Omega_2 \arrow[l, hook', "j_2"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T_{Z_1}X_1 \arrow[r, hook, "s_1"] \arrow[d, "T_Z f"'] \arrow[dr, phantom, "\square"] & \widetilde{X}_1 \arrow[d, "\widetilde{f}"] \\
T_{Z_2}X_2 \arrow[r, hook, "s_2"'] & \widetilde{X}_2\,,
\end{tikzcd}$$
in which the squares marked by □ are cartesian. Recall the adjunction isomorphism $f^!\,\mathrm{R}f_{!!} \simeq \mathrm{id}$. Hence it is enough to construct an isomorphism
$$\mathrm{R}f_{!!}\mathcal{L}_{\sigma_1}(Z_1, X_2) \xrightarrow{\sim} \mathrm{R}f_{!!}f^!\mathcal{L}_{\sigma_2}(Z_2, X_2).$$
Next recall that
$$\mathrm{R}f_{!!}f^!\mathcal{L}_{\sigma_2}(Z_2, X_2) \simeq \mathrm{R}\mathcal{IH}om\left(\mathbb{K}_{X_1}, \mathcal{L}_{\sigma_2}(Z_2, X_2)\right).$$
Therefore we may write:
$$\begin{aligned}
\mathrm{R}f_{!!}f^!\mathcal{L}_{\sigma_2}(Z_2,X_2) &\simeq \mathrm{R}\mathcal{IH}om\Big(\mathbb{K}_{X_1}, \mathrm{R}p_{2!!}\big(\mathbb{K}_{\overline{\Omega_2}} \otimes \beta_{\widetilde{X}_2}\big(\mathbb{K}_{P_{\sigma_2}} \otimes p_2^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}\big)\big)\Big) \\
&\simeq \mathrm{R}p_{2!!}\,\mathrm{R}\mathcal{IH}om\Big({p_2}^{-1}\,\mathbb{K}_{X_1}, \mathbb{K}_{\overline{\Omega_2}} \otimes \beta_{\widetilde{X}_2}\big(\mathbb{K}_{P_{\sigma_2}} \otimes p_2^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}\big)\Big) \\
&\simeq \mathrm{R}p_{2!!}\Big(\mathrm{R}\mathcal{IH}om\,(p_2^{-1}\,\mathbb{K}_{X_1}, \mathbb{K}_{\overline{\Omega_2}}) \otimes \beta_{\widetilde{X}_2}(\mathbb{K}_{P_{\sigma_2}} \otimes p_2^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2})\Big).
\end{aligned}$$
On the other hand, $P_{\sigma_1} = \widetilde{f}^{-1}P_{\sigma_2}$ implies
$$\begin{aligned}
&\mathrm{R}f_{!!}\mathcal{L}_{\sigma_1}(Z_1,X_1) \simeq \mathrm{R}f_{!!}\,\mathrm{R}p_{1!!}\Big(\mathbb{K}_{\overline{\Omega_1}} \otimes \beta_{\widetilde{X}_1}\big(\mathbb{K}_{P_{\sigma_1}} \otimes p_1^{-1}\,\mathrm{R}i_{1*}\omega^{\otimes -1}_{Z_1/X_1}\big)\Big) \\
&\quad \simeq \mathrm{R}p_{2!!}\,\mathrm{R}\widetilde{f}_{!!}\Big(\mathbb{K}_{\widetilde{f}^{-1}(\overline{\Omega_2})} \otimes \beta_{\widetilde{X}_1}\big(\widetilde{f}^{-1}\,\mathbb{K}_{P_{\sigma_1}} \otimes \widetilde{f}^{-1}p_2^{-1}\,\mathrm{R}s_{2*}\omega^{\otimes -1}_{Z_2/X_2} \otimes {p_1}^{-1}\omega_{X_1/X_2}\big)\Big) \\
&\quad \simeq \mathrm{R}p_{2!!}\,\mathrm{R}\widetilde{f}_{!!}\Big(\widetilde{f}^{-1}\Big(\mathbb{K}_{\overline{\Omega_2}} \otimes \beta_{\widetilde{X}_2}\big(\mathbb{K}_{P_{\sigma_2}} \otimes p_2^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}\big)\Big) \otimes \omega_{\widetilde{X}_1/\widetilde{X}_2}\Big) \\
&\quad \simeq \mathrm{R}p_{2!!}\Big(\mathbb{K}_{\overline{\Omega_2}} \otimes \beta_{\widetilde{X}_2}\big(\mathbb{K}_{P_{\sigma_2}} \otimes p_2^{-1}\,\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X_2}\big) \otimes \mathrm{R}\widetilde{f}_{!!}\omega_{\widetilde{X}_1/\widetilde{X}_2}\Big),
\end{aligned}$$
and it is enough to show that
$$\mathrm{R}\mathcal{IH}om(p_2^{-1}\,\mathbb{K}_{X_1}, \mathbb{K}_{\overline{\Omega_2}}) \simeq \mathbb{K}_{\overline{\Omega_2}} \otimes \mathrm{R}\widetilde{f}_{!!}\omega_{\widetilde{X}_1/\widetilde{X}_2}.$$
However we have the natural chain of isomorphisms
$$\begin{aligned}
\mathrm{R}\mathcal{IH}om(p_2^{-1}\,\mathbb{K}_{X_1}, \mathbb{K}_{\overline{\Omega_2}}) &\simeq \mathrm{R}\mathcal{IH}om(p_2^{-1}\,\mathbb{K}_{X_1}, \mathrm{R}j_{2*}\,\mathbb{K}_{\Omega_2}) \\
&\simeq \mathrm{R}j_{2*}\,\mathrm{R}\mathcal{IH}om({j_2}^{-1}p_2^{-1}\,\mathbb{K}_{X_1}, \mathbb{K}_{\Omega_2}) \simeq \mathrm{R}j_{2*}\,\mathrm{R}\mathcal{IH}om(\mathbb{K}_{\Omega_1}, \mathbb{K}_{\Omega_2}).
\end{aligned}$$

On the other hand, we have, as an object of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_{\Omega_2}))$,

$$\mathrm{R}\mathcal{I}\mathcal{H}om(\mathbb{K}_{\Omega_1}, \mathbb{K}_{\Omega_2}) \simeq j_2^{-1}\ \mathrm{R}\widetilde{f}_*\omega_{\widetilde{X}_1/\widetilde{X}_2},$$

and hence

$$\begin{aligned}\mathrm{R}\mathcal{I}\mathcal{H}om(p_2^{-1}\,\mathbb{K}_{X_1}, \mathbb{K}_{\overline{\Omega_2}}) &\simeq \mathrm{R}j_{2*}j_2^{-1}\,\mathrm{R}\widetilde{f}_*\omega_{\widetilde{X}_1/\widetilde{X}_2}\\ &\simeq \mathrm{R}j_{2*}\,\mathbb{K}_{\Omega_1}\otimes\mathrm{R}\widetilde{f}_*\omega_{\widetilde{X}_1/\widetilde{X}_2} \simeq \mathbb{K}_{\overline{\Omega_1}}\otimes\mathrm{R}\widetilde{f}_{!!}\omega_{\widetilde{X}_1/\widetilde{X}_2}.\end{aligned}$$

□

Proposition 1.3.8. *Let (X, Z_1, σ_1) and (X, Z_2, σ_2) be kernel data on the same base manifold X. Assume that Z_1, Z_2 are transversal submanifolds. Then there is a natural morphism*

$$\mathcal{L}_{\sigma_1}(Z_1, X)\otimes\mathcal{L}_{\sigma_2}(Z_2, X)\longrightarrow\mathcal{L}_{\sigma_1+\sigma_2}(Z_1\cap Z_2, X)\otimes\widetilde{\mathbb{K}}_{\mathcal{T}(\sigma_1)\cap\mathcal{T}(\sigma_2)}\,.$$

Proof. Set $Z = Z_1\cap Z_2$, $\sigma = \sigma_1+\sigma_2$ and $N = \mathcal{T}(\sigma_1)\cap\mathcal{T}(\sigma_2)\subset\mathcal{T}(\sigma)\subset Z$.

(i) Assume first that $\sigma_1(x)$ and $\sigma_2(x)$ are linearly independent vectors of T^*X for every $x\in Z$. Then we have

$$\mathcal{L}_{\sigma_k}(Z_k, X)\otimes\widetilde{\mathbb{K}}_N \simeq \text{“}\varinjlim_{U_k}\text{”}\,\mathbb{K}_{U_k}\otimes\widetilde{\mathbb{K}}_N\otimes\beta_X\left(\mathrm{R}i_{k*}\omega^{\otimes -1}_{Z_k/X}\right),$$

where the inductive limits is taken over the family of open subsets U_k of X such that $\mathrm{C}_{Z_k}(U_k)\cap P_{\sigma_k}\subset Z_k$. For such open subsets U_1, U_2, we have

$$\mathrm{C}_Z(U_1\cap U_2)\cap\left(P_\sigma\underset{Z}{\times}N\right)\subset Z,$$

since $P_\sigma\underset{Z}{\times}N\subset P_{\sigma_1}\cup P_{\sigma_2}$. Hence we get a natural morphism

$$\begin{aligned}&\mathcal{L}_{\sigma_1}(Z_1, X)\otimes\mathcal{L}_{\sigma_2}(Z_2, X)\otimes\widetilde{\mathbb{K}}_N\\ &\quad\simeq\left(\text{“}\varinjlim_{U_1}\text{”}\,\mathbb{K}_{U_1}\otimes\beta\left(\mathrm{R}i_{1*}\omega^{\otimes -1}_{Z_1/X}\right)\right)\otimes\left(\text{“}\varinjlim_{U_2}\text{”}\,\mathbb{K}_{U_2}\otimes\beta\left(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X}\right)\right)\otimes\widetilde{\mathbb{K}}_N\\ &\quad\to\left(\text{“}\varinjlim_{U}\text{”}\,\mathbb{K}_U\right)\otimes\beta\left(\mathrm{R}i_{1*}\omega^{\otimes -1}_{Z_1/X}\right)\otimes\beta\left(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X}\right)\otimes\widetilde{\mathbb{K}}_N,\end{aligned}$$

where U ranges over the family of open subsets of X such that $\mathrm{C}_Z(U)\cap\left(P_\sigma\underset{Z}{\times}N\right)\subset Z$. Since Z_1 and Z_2 are transversal submanifolds of X, we have $\omega^{\otimes -1}_{Z/X}\simeq\left(\omega^{\otimes -1}_{Z_1/X}|_Z\right)\otimes\left(\omega^{\otimes -1}_{Z_2/X}|_Z\right)$. Hence we obtain

$$\text{“}\varinjlim_{U}\text{”}\,\mathbb{K}_U\otimes\beta_X\left(\mathrm{R}i_{1*}\omega^{\otimes -1}_{Z_1/X}\right)\otimes\beta_X\left(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X}\right)\otimes\widetilde{\mathbb{K}}_N\simeq\mathcal{L}_\sigma(Z, X)\otimes\widetilde{\mathbb{K}}_N,$$

which provides the desired morphism.

(ii) Consider the general case. We set $\mathbb{A}^n_X = X\times\mathbb{R}^n$ for $n = 1, 2$. We use coordinates (x, t_1, t_2) on $\mathbb{A}^2_X$. We regard the manifold $\mathbb{A}^1_{Z_k}$ as a submanifold of $\mathbb{A}^2_X$ by

$$\mathbb{A}^1_{Z_k} := \{(x, t_1, t_2)\,;\, x\in Z_k,\ t_k = 0\}\,,$$

and $\mathbb{A}^1_X$ as the submanifold $\{t_2 = 0\}$ of $\mathbb{A}^2_X$. We identify Z with

$$\mathbb{A}^1_{Z_1}\cap\mathbb{A}^1_{Z_2} = \{(x, t_1, t_2)\,;\, x\in Z,\ t_1 = t_2 = 0\}\,.$$

Thus we obtain the following commutative diagrams

$$\begin{tikzcd} & X \arrow[r, hook, "i"] & \mathbb{A}^1_X \arrow[r, hook, "i'"] & \mathbb{A}^2_X \\ Z \arrow[r, hook] & Z_1 \arrow[u, hook] \arrow[r, "\sim"] & Z_1 \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\mathrm{tr}"] & \mathbb{A}^1_{Z_1} \arrow[u, hook, "j_1"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X \arrow[r, hook, "i"] & \mathbb{A}^1_X \arrow[r, hook, "i'"] & \mathbb{A}^2_X \\ Z \arrow[r, hook] & Z_2 \arrow[u, hook] \arrow[r, hook] \arrow[ur, phantom, "\mathrm{tr}"] & \mathbb{A}^1_{Z_2} \arrow[u, hook] \arrow[r, "\sim"] & \mathbb{A}^1_{Z_2} \arrow[u, hook, "j_2"'] \end{tikzcd}$$

where $j_1(z_1, t) = (z_1, 0, t)$ and $j_2(z_2, t) = (z_2, t, 0)$. Note that the squares marked with tr are transversal. Define the sections

$$\begin{aligned} \tilde{\sigma}_1 = \sigma_1 + dt_1 &\ :\ \mathbb{A}^1_{Z_1} \longrightarrow T^*\mathbb{A}^2_X, \\ \tilde{\sigma}_2 = \sigma_2 + dt_2 &\ :\ \mathbb{A}^1_{Z_2} \longrightarrow T^*\mathbb{A}^2_X, \\ \tilde{\sigma} = \sigma_1 + \sigma_2 + dt_1 + dt_2 &\ :\ Z \longrightarrow T^*\mathbb{A}^2_X. \end{aligned}$$

Clearly $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are linearly independent at each point, and the result in the first part gives a morphism

$$\mathcal{L}_{\tilde{\sigma}_1}\left(\mathbb{A}^1_{Z_1}, \mathbb{A}^2_X\right) \otimes \mathcal{L}_{\tilde{\sigma}_2}\left(\mathbb{A}^1_{Z_2}, \mathbb{A}^2_X\right) \longrightarrow \mathcal{L}_{\tilde{\sigma}}(Z, \mathbb{A}^2_X) \otimes \widetilde{\mathbb{K}}_N .$$

We then deduce morphisms with the help of Proposition 1.3.3 and Proposition 1.3.7

$$\begin{aligned} \mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X) &\simeq i^!\mathcal{L}_{\tilde{\sigma}_1}\left(Z_1, \mathbb{A}^1_X\right) \otimes i^{-1}\mathcal{L}_{\sigma_2}\left(\mathbb{A}^1_{Z_2}, \mathbb{A}^1_X\right) \\ &\to i^!\left(\mathcal{L}_{\tilde{\sigma}_1}\left(Z_1, \mathbb{A}^1_X\right) \otimes \mathcal{L}_{\sigma_2}\left(\mathbb{A}^1_{Z_2}, \mathbb{A}^1_X\right)\right) \\ &\simeq i^!\left(i'^{-1}\mathcal{L}_{\tilde{\sigma}_1}\left(\mathbb{A}^1_{Z_1}, \mathbb{A}^2_X\right) \otimes i'^!\mathcal{L}_{\tilde{\sigma}_2}\left(\mathbb{A}^1_{Z_2}, \mathbb{A}^2_X\right)\right) \\ &\to i^!i'^!\left(\mathcal{L}_{\tilde{\sigma}_1}\left(\mathbb{A}^1_{Z_1}, \mathbb{A}^2_X\right) \otimes \mathcal{L}_{\tilde{\sigma}_2}\left(\mathbb{A}^1_{Z_2}, \mathbb{A}^2_X\right)\right) \\ &\to i^!i'^!\big(\mathcal{L}_{\tilde{\sigma}}(Z, \mathbb{A}^2_X) \otimes \widetilde{\mathbb{K}}_N\big) \simeq \mathcal{L}_{\sigma}(Z, X) \otimes \widetilde{\mathbb{K}}_N, \end{aligned}$$

which completes the proof. □

Remark 1.3.9. Although we do not give proofs, the following two facts hold.

(i) If σ_1 and σ_2 are linearly independent, the two morphisms constructed in the parts (i) and (ii) of the proof of Proposition 1.3.8 coincide.

(ii) If (X, Z_3, σ_3) is a third kernel data such that (Z_1, Z_2), (Z_1, Z_3) and (Z_2, Z_3) are transversal in X and that $(Z_1 \cap Z_3, Z_2 \cap Z_3)$ is transversal in Z_3, then the following diagram is commutative where $N = \mathfrak{T}(\sigma_1) \cap \mathfrak{T}(\sigma_2) \cap \mathfrak{T}(\sigma_3)$:

$$\begin{tikzcd} \mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \mathcal{L}_{\sigma_3}(Z_3, X) \arrow[r] \arrow[d] & \mathcal{L}_{\sigma_1+\sigma_2}(Z_1 \cap Z_2, X) \otimes \mathcal{L}_{\sigma_3}(Z_3, X) \otimes \widetilde{\mathbb{K}}_N \arrow[d] \\ \mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2+\sigma_3}(Z_2 \cap Z_3, X) \otimes \widetilde{\mathbb{K}}_N \arrow[r] & \mathcal{L}_{\sigma_1+\sigma_2+\sigma_3}(Z_1 \cap Z_2 \cap Z_3, X) \otimes \widetilde{\mathbb{K}}_N, \end{tikzcd}$$

i.e. the composition morphisms are associative.

Lemma 1.3.10. *Let (X, Z_1, σ_1), (X, Z_2, σ_2) be kernel data on X and assume that Z_1, Z_2 are transversal submanifolds of X and that σ_1 and σ_2 never vanish. Let $f\colon X \to Y$ be a smooth morphism which induces a closed embedding $Z_1 \cap Z_2 \hookrightarrow Y$. Assume the following condition:*

$$\big(\mathbb{R}_{\geq 0}\sigma_1(x) + \mathbb{R}_{\geq 0}\sigma_2(x)\big) \cap T^*_{f(x)}Y = \{0\} \textit{ for every } x \in \mathfrak{T}(\sigma_1) \cap \mathfrak{T}(\sigma_2).$$

Here $T^*_{f(x)}Y$ is regarded as a subspace of T^*_xX by f_d. Then we have

$$\mathrm{R}f_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X)\right) \simeq 0.$$

Proof. Let us show that

$$\mathrm{R}f_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1,X)\otimes\mathcal{L}_{\sigma_2}(Z_2,X)\otimes\widetilde{\mathbb{K}}_{x_0}\right)\simeq 0$$

for any $x_0\in\mathfrak{T}(\sigma_1)\cap\mathfrak{T}(\sigma_2)$. We first reduce the proof to the case where X is of relative dimension one over Y. Assume the assertion to be true for relative one-dimensional morphisms. Set $E=T_{x_0}(f^{-1}f(x_0))$. Then by the assumption, E satisfies $\big(\mathbb{R}_{\geq 0}\sigma_1(x_0)+\mathbb{R}_{\geq 0}\sigma_2(x_0)\big)\cap E^{\perp}=\{0\}$. Hence there exists a line $\ell\subset E$ such that $\big(\mathbb{R}_{\geq 0}\sigma_1(x_0)+\mathbb{R}_{\geq 0}\sigma_2(x_0)\big)\cap \ell^{\perp}=\{0\}$. Decompose f into the composition of smooth morphisms $X\xrightarrow{g} Y'\xrightarrow{h} Y$ on a neighborhood of x_0 such that g and h are smooth and $T_{x_0}(g^{-1}g(x_0))=\ell$. Then g satisfies the conditions in the lemma. Hence applying to g the relative one-dimensional morphism case, we obtain $\mathrm{R}g_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1,X)\otimes\mathcal{L}_{\sigma_2}(Z_2,X)\otimes\widetilde{\mathbb{K}}_{x_0}\right)\simeq 0$, which implies the desired result.

Now assume that f has relative dimension one. Since $\sigma_k(x_0)\notin T^*_{f(x_0)}Y$, the map $Z_k\to Y$ is a (local) embedding, and $T_{x_0}Z_k=f_*^{-1}\big(T_{f(x_0)}Z'_k\big)\cap\sigma_k(x_0)^{-1}(0)$, where $Z'_k:=f(Z_k)\subset Y$. Then Z'_1 and Z'_2 are transversal, and $f(Z_1\cap Z_2)$ is a hypersurface of $Z'_1\cap Z'_2$ since

$$\begin{aligned}\operatorname{codim}_Y(f(Z_1\cap Z_2))&=\operatorname{codim}_X(Z_1\cap Z_2)-1=\operatorname{codim}_X(Z_1)+\operatorname{codim}_X(Z_2)-1\\&=\operatorname{codim}_Y(Z'_1)+\operatorname{codim}_Y(Z'_2)+1=\operatorname{codim}_Y(Z'_1\cap Z'_2)+1.\end{aligned}$$

Since $T_{x_0}(Z_1\cap Z_2)=f_*^{-1}\big(T_{f(x_0)}(Z'_1\cap Z'_2)\big)\cap\sigma_1(x_0)^{-1}(0)\cap\sigma_2(x_0)^{-1}(0)$, the vectors $\sigma_1(x_0)$ and $\sigma_2(x_0)$ are linearly independent. By multiplying by a positive constant, we may therefore assume that

$$\sigma_1(x_0)-\sigma_2(x_0)\in T^*_{f(x_0)}Y\setminus\{0\}.$$

Take a local coordinate system (t,y_1,y_2,z) of Y such that

$$Z'_k=\{y_k=0\}\text{ and }\sigma_2(x_0)-\sigma_1(x_0)=dt.$$

Then take a local coordinate system (x,t,y_1,y_2,z) of X such that $\sigma_1(x_0)=-dx$ (and hence $\sigma_2(x_0)=dt-dx$), and $Z_1=\{y_1=0,\,x=0\}$ and f is given by forgetting x. Set $Z_2=\{y_2=0,\,x=\varphi(t,y_1,z)\}$. Then replacing $\varphi(t,y_1,z)$ with t, we may assume from the beginning that

$$Z_2=\{y_2=0,\,x=t\},\ Z_1\cap Z_2=\{y_1=0,\,y_2=0,\,x=t=0\}.$$

Then we have

$$\mathcal{L}_{\sigma_1}(Z_1,X)\otimes\mathcal{L}_{\sigma_2}(Z_2,X)\otimes\widetilde{\mathbb{K}}_{x_0}\simeq\text{“}\varinjlim_{\delta>0,\,\varepsilon>0}\text{”}\left(\mathbb{K}_{U^1_{\delta,\varepsilon}}\otimes\mathbb{K}_{U^2_{\delta,\varepsilon}}\right)\otimes\beta_X\big(\mathrm{R}i_{1*}\omega^{\otimes-1}_{Z_1/X}\otimes\mathrm{R}i_{2*}\omega^{\otimes-1}_{Z_2/X}\big)\otimes\widetilde{\mathbb{K}}_{x_0},$$

where the open sets $U^k_{\delta,\varepsilon}$ are given by

$$U^1_{\delta,\varepsilon}=\{\varepsilon|y_1|<x\leq\delta\}\quad\text{and}\quad U^2_{\delta,\varepsilon}=\{\varepsilon|y_2|<x-t\leq\delta\}.$$

Hence we have

$$U^1_{\delta,\varepsilon}\cap U^2_{\delta,\varepsilon}=\{\max(\varepsilon|y_1|,\,\varepsilon|y_2|+t)<x\leq\min(\delta,\,\delta+t)\}.$$

Then the result follows from

$$\mathrm{R}f_!(\mathbb{K}_{U^1_{\delta,\varepsilon}\cap U^2_{\delta,\varepsilon}})\simeq 0.$$

∎

Proposition 1.3.11. *Let (X, Z_1, σ_1), (X, Z_2, σ_2) be kernel data on X and (Y, Z, σ) a kernel data on Y. Assume that Z_1, Z_2 are transversal submanifolds of X. Let $f\colon X \to Y$ be a smooth morphism which induces an isomorphism $Z_1 \cap Z_2 \xrightarrow{\sim} Z$. Let N be a closed subset of $\mathcal{T}(\sigma_1) \cap \mathcal{T}(\sigma_2)$ satisfying the following conditions:*

(i) *$\mathcal{Z}(\sigma_1) \cap \mathcal{Z}(\sigma_2) \subset N$,*
(ii) *$f^*\sigma(x) = \sigma_1(x) + \sigma_2(x)$ for every $x \in N$,*
(iii) *$\sigma_1(x) \notin T^*_{f(x)}Y$ for any $x \in N \setminus \big(\mathcal{Z}(\sigma_1) \cup \mathcal{Z}(\sigma_2)\big)$,*
(iv) *$\big(\mathbb{R}_{\geq 0}\sigma_1(x) + \mathbb{R}_{\geq 0}\sigma_2(x)\big) \cap T^*_{f(x)}Y = \{0\}$ for every $x \in \big(\mathcal{T}(\sigma_1) \cap \mathcal{T}(\sigma_2)\big) \setminus N$,*
(v) *the morphism $Z_k \to Y$ is smooth at each point of $\mathcal{Z}(\sigma_k)$ for $k = 1, 2$.*

Then there is a natural isomorphism

$$\mathrm{R}f_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X)\right) \xrightarrow{\sim} \mathcal{L}_{\sigma}(Z, Y) \otimes \widetilde{\mathbb{K}}_{f(N)}\,.$$

Proof. The morphism is obtained as the composition

$$\begin{aligned}\mathrm{R}f_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X)\right) &\to \mathrm{R}f_{!!}\big(\mathcal{L}_{\sigma_1+\sigma_2}(Z_1 \cap Z_2, X) \otimes \widetilde{\mathbb{K}}_N\big)\\ &\simeq \mathrm{R}f_{!!}\big(\mathcal{L}_{f^*\sigma}(Z_1 \cap Z_2, X) \otimes \widetilde{\mathbb{K}}_N\big) \to \mathcal{L}_{\sigma}(Z, Y) \otimes \widetilde{\mathbb{K}}_{f(N)}\,.\end{aligned}$$

In order to see that it is an isomorphism, it is enough to prove the isomorphism

$$\mathrm{R}f_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \widetilde{\mathbb{K}}_{x_0}\right) \xrightarrow{\sim} \mathcal{L}_{\sigma}(Z, Y) \otimes \widetilde{\mathbb{K}}_{f(N)} \otimes \widetilde{\mathbb{K}}_{f(x_0)}$$

for any $x_0 \in \mathcal{T}(\sigma_1) \cap \mathcal{T}(\sigma_2)$.

(a) Assume first that $\sigma_1(x_0) = \sigma_2(x_0) = 0$. Then, (i) implies $x_0 \in N$, and we have $\sigma(f(x_0)) = 0$ by (ii). Hence Proposition 1.2.8 implies

$$\begin{aligned}\mathrm{R}f_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \widetilde{\mathbb{K}}_{x_0}\right) &\simeq \mathrm{R}f_{!!}\left(\mathbb{K}_{Z_1} \otimes \mathbb{K}_{Z_2} \otimes \widetilde{\mathbb{K}}_{x_0}\right)\\ &\simeq \mathbb{K}_Z \otimes \widetilde{\mathbb{K}}_{f(x_0)} \simeq \mathcal{L}_{\sigma}(Z, Y) \otimes \widetilde{\mathbb{K}}_{f(N)} \otimes \widetilde{\mathbb{K}}_{f(x_0)}\,.\end{aligned}$$

(b) Assume $\sigma_1(x_0) = 0$ and $\sigma_2(x_0) \neq 0$. Then we have

$$\begin{aligned}\mathrm{R}f_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \widetilde{\mathbb{K}}_{x_0}\right) &\simeq \mathrm{R}f_{!!}\left(\mathbb{K}_{Z_1} \otimes \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \widetilde{\mathbb{K}}_{x_0}\right)\\ &\simeq \mathrm{R}f_{!!} i_{1!!} i_1^{-1} \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \widetilde{\mathbb{K}}_{f(x_0)},\end{aligned}$$

where $i_1\colon Z_1 \to X$ is the inclusion. Proposition 1.3.3 implies $i_1^{-1}\mathcal{L}_{\sigma_2}(Z_2, X) \simeq \mathcal{L}_{\sigma_2}(Z_1 \cap Z_2, Z_1)$. Note that $Z_1 \to Y$ is smooth at x_0 by the assumption (v). If $x_0 \in N$, then Corollary 1.3.5, along with by the hypothesis (ii), implies $\mathrm{R}f_{!!} i_{1!!}\mathcal{L}_{\sigma_2}(Z_1 \cap Z_2, Z_1) \simeq \mathcal{L}_{\sigma}(Z, Y)$. Assume $x \in \big(\mathcal{T}(\sigma_1) \cap \mathcal{T}(\sigma_2)\big) \setminus N$. Then (iv) implies that $\sigma_2(x_0) \notin T^*{}_f(x_0)Y$, and hence Lemma 1.3.6 implies $\mathrm{R}f_{!!} i_{1!!}\mathcal{L}_{\sigma_2}(Z_1 \cap Z_2, Z_1) \simeq 0$.

(c) Therefore we may assume that $\sigma_1(x_0) \neq 0$ and $\sigma_2(x_0) \neq 0$. If $x_0 \notin N$, then the result follows from (iv) and Lemma 1.3.10. We may assume therefore $x_0 \in N$. Similarly to the proof of Lemma 1.3.10, we first reduce the proof to the case where X is of relative dimension one over Y. Assume the theorem to be true in the relative one-dimensional morphism case. Set $E = T_{x_0}(f^{-1}f(x_0))$. Let us choose a line $\ell \subset E$ such that $\sigma_1(x_0)|_\ell \neq 0$, and then decompose f into $X \xrightarrow{g} Y' \xrightarrow{h} Y$ on a neighborhood of x_0 such that g and h are smooth, and $T_{x_0}(g^{-1}g(x_0)) = \ell$. Then g satisfies the conditions (i)–(iv), and applying the relative dimension one case to g, we obtain

$$\mathrm{R}f_{!!}\left(\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X)\right) \simeq \mathrm{R}h_{!!}\mathcal{L}_{h^*\sigma}\left(g(Z_1 \cap Z_2), Y'\right) \simeq \mathcal{L}_{\sigma}(Z, Y),$$

where the last isomorphism is deduced from Corollary 1.3.5.

Hence me may assume that the relative dimension of X over Y is one. By the assumption (iii), $Z_k \to Y$ is a (local) embedding and $T_{x_0}Z_k = f_*^{-1}\big(T_{f(x_0)}Z'_k\big) \cap \sigma_k(x_0)^{-1}(0)$ where $Z'_k := f(Z_k)$. Then Z'_1 and Z'_2 are transversal submanifolds of Y and Z is a one-codimensional submanifold of $Z' := Z'_1 \cap Z'_2$. We have

$$\sigma(f(x_0)) \notin T^*_{Z'}Y.$$

Indeed, we have

$$\begin{aligned} T_{x_0}(Z_1 \cap Z_2) &= f_*^{-1}\big(T_{f(x_0)}Z'\big) \cap \sigma_1(x_0)^{-1}(0) \cap \sigma_2(x_0)^{-1}(0) \\ &= f_*^{-1}\Big(T_{f(x_0)}Z' \cap \sigma(f(x_0))^{-1}(0)\Big) \cap \sigma_1(x_0)^{-1}(0), \end{aligned}$$

which implies $T_{f(x_0)}Z = T_{f(x_0)}Z' \cap \sigma(f(x_0))^{-1}(0) \neq T_{f(x_0)}Z'$.

Hence we can take local coordinates $(t, y_1, y_2, z) \in \mathbb{R} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^n$ of Y such that $\sigma(f(x_0)) = -dt(f(x_0))$ and $Z'_k = \{y_k = 0\}$ $(k = 1, 2)$. Then we can choose a system of coordinates (x, t, y_1, y_2, z) on X such that f is given by forgetting x, $\sigma_1(x_0) = -dx(x_0)$ by (iii) (and hence $\sigma_2(x_0) = dx(x_0) - dt(x_0)$) and that $Z_1 = \{y_1 = 0, x = 0\}$. Set $Z_2 = \{y_2 = 0, x = \varphi(t, y_1, z)\}$. Replacing $\varphi(t, y_1, z)$ with t, we may assume from the beginning that

$$Z_2 = \{y_2 = 0,\ x = t\} \text{ and } Z = \{y_1 = 0,\ y_2 = 0,\ t = 0\}.$$

We have then using Corollary 1.2.13

$$\begin{aligned} \mathcal{L}_{\sigma_1}(Z_1, X) \otimes \widetilde{\mathbb{K}}_{x_0} &\simeq \widetilde{\mathbb{K}}_{x_0} \otimes \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{K}_{U^1_\varepsilon} \otimes \beta_X\big(\mathrm{R}i_{1*}\omega^{\otimes -1}_{Z_1/X}\big), \\ \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \widetilde{\mathbb{K}}_{x_0} &\simeq \widetilde{\mathbb{K}}_{x_0} \otimes \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{K}_{U^2_\varepsilon} \otimes \beta_X\big(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X}\big), \end{aligned}$$

where the open sets U^k_ε are given by

$$U^1_\varepsilon = \{\varepsilon|y_1| < x\} \quad \text{and} \quad U^2_\varepsilon = \{\varepsilon|y_2| < t - x\}.$$

We may therefore write

$$\begin{aligned} &\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \widetilde{\mathbb{K}}_{x_0} \\ &\qquad \simeq \widetilde{\mathbb{K}}_{x_0} \otimes \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{K}_{U^1_\varepsilon \cap U^2_\varepsilon} \otimes \beta_X\big(\mathrm{R}i_{1*}\omega^{\otimes -1}_{Z_1/X}\big) \otimes \beta_X\big(\mathrm{R}i_{2*}\omega^{\otimes -1}_{Z_2/X}\big) \\ &\qquad \simeq \widetilde{\mathbb{K}}_{x_0} \otimes \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{K}_{U^1_\varepsilon \cap U^2_\varepsilon} \otimes \beta_X\big(f^{-1}\,\mathrm{R}i_*\omega^{\otimes -1}_{Z/Y}\big) \otimes \omega^{\otimes -1}_{X/Y}. \end{aligned}$$

Since the relative dimension of X over Y is one, we have $\omega^{\otimes -1}_{X/Y} \otimes \widetilde{\mathbb{K}}_{x_0} \simeq \widetilde{\mathbb{K}}_{x_0}[1]$, and we deduce an isomorphism

$$\begin{aligned} &\mathrm{R}f_{!!}\Big(\mathcal{L}_{\sigma_1}(Z_1, X) \otimes \mathcal{L}_{\sigma_2}(Z_2, X) \otimes \widetilde{\mathbb{K}}_{x_0}\Big) \\ &\qquad \simeq \mathrm{R}f_{!!}\Big(\widetilde{\mathbb{K}}_{x_0} \otimes \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{K}_{U^1_\varepsilon \cap U^2_\varepsilon} \otimes \omega_{X/Y}\Big) \otimes \beta_Y\, \mathrm{R}i_*\omega^{\otimes -1}_{Z/Y} \\ &\qquad \simeq \mathrm{R}f_{!!}\Big(\text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{K}_{U^1_\varepsilon \cap U^2_\varepsilon}\Big)[1] \otimes \widetilde{\mathbb{K}}_{f(x_0)} \otimes \beta_Y\, \mathrm{R}i_*\omega^{\otimes -1}_{Z/Y}. \end{aligned}$$

Since $U^1_\varepsilon \cap U^2_\varepsilon = \{\varepsilon|y_1| < x < t - \varepsilon|y_2|\}$, we have

$$\mathrm{R}f_!\big(\mathbb{K}_{U^1_\varepsilon \cap U^2_\varepsilon}\big) \simeq \mathbb{K}_{\{\varepsilon(|y_1|+|y_2|)<t\}}[-1].$$

Hence we finally deduce that

$$\begin{aligned}\mathrm{R}f_{!!}\Big(\mathcal{L}_{\sigma_1}(Z_1,X)\otimes\mathcal{L}_{\sigma_2}(Z_2,X)\otimes\widetilde{\mathbb{K}}_{x_0}\Big) &\simeq \Big(\text{“}\varinjlim_{\varepsilon>0}\text{”}\,\mathbb{K}_{\{\varepsilon(|y_1|+|y_2|)<t\}}\Big)\otimes\beta_Y\,\mathrm{R}i_*\omega_{Z/Y}^{\otimes-1}\otimes\widetilde{\mathbb{K}}_{f(x_0)}\\ &\simeq \mathcal{L}_\sigma(Z,Y)\otimes\widetilde{\mathbb{K}}_{f(x_0)}\,.\end{aligned}$$

□

Proposition 1.3.12. *Let (X_1,X_2,X_3) be a triplet of manifolds and $(X_i\times X_j,Z_{ij},\sigma_{ij})$ be a kernel data for $1\le i<j\le 3$. Assume that $Z_{12}\times X_3$ and $X_1\times Z_{23}$ are transversal in $X_1\times X_2\times X_3$ and that the projections $p_{ij}\colon X_1\times X_2\times X_3\to X_i\times X_j$ induce an isomorphism $Z_{12}\underset{X_2}{\times}Z_{23}\xrightarrow{\sim}Z_{13}$. Let us denote by $p_2\colon X_1\times X_2\times X_3\to X_2$ the second projection and by $p_{2*}\colon T^*(X_1\times X_2\times X_3)\to T^*X_2$ the induced projection. Let $N\subset\mathcal{T}(\sigma_{12})\underset{X_2}{\times}\mathcal{T}(\sigma_{23})$ be a closed subset satisfying the following conditions:*

(i) *$\mathcal{Z}(\sigma_{12})\underset{X_2}{\times}\mathcal{Z}(\sigma_{23})\subset N$,*
(ii) *$p_{13}^*\sigma_{13}(x)=p_{12}^*\sigma_{12}(x)+p_{23}^*\sigma_{23}(x)$ for every $x\in N$,*
(iii) *$p_{2*}\sigma_{12}(x)\notin T^*_{X_2}X_2$ for any $x\in N\setminus\big(\mathcal{Z}(\sigma_{12})\times X_3\cup X_1\times\mathcal{Z}(\sigma_{23})\big)$,*
(iv) *$\mathbb{R}_{\ge0}p_{2*}\sigma_{12}(x)\ne\mathbb{R}_{\le0}p_{2*}\sigma_{23}(x)$ for every $x\in\big(\mathcal{T}(\sigma_{12})\underset{X_2}{\times}\mathcal{T}(\sigma_{23})\big)\setminus N$,*
(v) *the morphism $Z_{12}\to X_1$ is smooth at each point of $\mathcal{Z}(\sigma_{12})$ and the morphism $Z_{23}\to X_3$ is smooth at each point of $\mathcal{Z}(\sigma_{23})$.*

Then we have an isomorphism

$$\mathcal{L}_{\sigma_{12}}(Z_{12},X_1\times X_2)\circ\mathcal{L}_{\sigma_{23}}(Z_{23},X_2\times X_3)\xrightarrow{\sim}\mathcal{L}_{\sigma_{13}}(Z_{13},X_1\times X_3)\otimes\widetilde{\mathbb{K}}_{f(N)}\,.$$

Proof. By Proposition 1.3.3, we have

$$\begin{aligned}{p_{12}}^{-1}\mathcal{L}_{\sigma_{12}}(Z_{12},X_1\times X_2)&\simeq\mathcal{L}_{p_{12}^*\sigma_{12}}(Z_{12}\times X_3,X_1\times X_2\times X_3),\\ {p_{23}}^{-1}\mathcal{L}_{\sigma_{23}}(Z_{23},X_2\times X_3)&\simeq\mathcal{L}_{p_{23}^*\sigma_{23}}(X_1\times Z_{23},X_1\times X_2\times X_3),\end{aligned}$$

and Proposition 1.3.11 implies

$$\begin{aligned}&\mathrm{R}p_{13!!}\left(\mathcal{L}_{p_{12}^*\sigma_{12}}(Z_{12}\times X_3,X_1\times X_2\times X_3)\otimes\mathcal{L}_{p_{23}^*\sigma_{23}}(X_1\times Z_{23},X_1\times X_2\times X_3)\right)\\ &\qquad\simeq\mathcal{L}_{\sigma_{13}}(Z_{13},X_1\times X_3)\otimes\widetilde{\mathbb{K}}_{f(N)}\,.\end{aligned}$$

□

2. Microlocalization of ind-sheaves

2.1. **The kernel $\mathrm{K}_{\mathfrak{X}}$ of ind-microlocalization.** We shall construct the kernel of microlocalization by the methods of the preceding section using the fundamental 1-form ω_X of T^*X. Since the construction uses only a 1-form, we shall discuss it on homogeneous symplectic manifolds. A *homogeneous symplectic manifold* is a manifold $\mathfrak{X}$ of even dimension endowed with a 1-form $\omega_{\mathfrak{X}}$ such that $(d\,\omega_{\mathfrak{X}})^{\dim\mathfrak{X}/2}$ never vanishes. It is a classical result that there locally exists a coordinate system $(x_1,\dots,x_n;\xi_1,\dots,\xi_n)$ where $\omega_{\mathfrak{X}}$ does not vanish and

$$\omega_{\mathfrak{X}}=\sum_{i=1}^n\xi_i dx_i. \tag{2.1}$$

Let $p_i\colon\mathfrak{X}\times\mathfrak{X}\to\mathfrak{X}$ $(i=1,2)$ be the projection and let $\Delta_{\mathfrak{X}}$ denote the diagonal of $\mathfrak{X}\times\mathfrak{X}$. Then $\sigma_{\mathfrak{X}}=p_1^*\,\omega_{\mathfrak{X}}-p_2^*\,\omega_{\mathfrak{X}}$ gives a section of $T^*_{\Delta_{\mathfrak{X}}}(\mathfrak{X}\times\mathfrak{X})\to\Delta_{\mathfrak{X}}$.

Definition 2.1.1. *The microlocalization kernel is the kernel defined on $\mathfrak{X} \times \mathfrak{X}$ by:*

$$\mathrm{K}_{\mathfrak{X}} = \mathcal{L}_{\sigma_{\mathfrak{X}}}(\Delta_{\mathfrak{X}}, \mathfrak{X} \times \mathfrak{X}) \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_{\mathfrak{X}\times\mathfrak{X}})).$$

Lemma 2.1.2. *There is a natural morphism*

$$\varepsilon_{\mathfrak{X}} \colon \mathbb{K}_{\Delta_{\mathfrak{X}}} \longrightarrow \mathrm{K}_{\mathfrak{X}}$$

such that the compositions

$$\mathrm{K}_{\mathfrak{X}} \simeq \mathrm{K}_{\mathfrak{X}} \circ \mathbb{K}_{\Delta_{\mathfrak{X}}} \xrightarrow{\mathrm{K}_{\mathfrak{X}} \circ \varepsilon_{\mathfrak{X}}} \mathrm{K}_{\mathfrak{X}} \circ \mathrm{K}_{\mathfrak{X}},$$
$$\mathrm{K}_{\mathfrak{X}} \simeq \mathbb{K}_{\Delta_{\mathfrak{X}}} \circ \mathrm{K}_{\mathfrak{X}} \xrightarrow{\varepsilon_{\mathfrak{X}} \circ \mathrm{K}_{\mathfrak{X}}} \mathrm{K}_{\mathfrak{X}} \circ \mathrm{K}_{\mathfrak{X}}$$

are isomorphisms, and these two isomorphisms coincide.

Proof. We have constructed the morphism $\varepsilon_{\mathfrak{X}}$ in Corollary 1.2.10. The second statement easily follows from Proposition 1.3.12. The last statement follows from Lemma 2.1.3 below □

Lemma 2.1.3. *Let $F \colon \mathcal{C} \to \mathcal{C}$ be a functor and $\alpha \colon \mathrm{id}_{\mathcal{C}} \to F$ a morphism of functors. Assume that for any object $X \in \mathrm{Ob}(\mathcal{C})$ the morphisms*

$$\alpha_{F(X)} \colon F(X) \to F(F(X)) \qquad F(\alpha_X) \colon F(X) \to F(F(X))$$

are isomorphisms. Then

(i) *For any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the composition with α_X defines a bijection*

$$\mathrm{Hom}_{\mathcal{C}}(F(X), F(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, F(Y)),$$

(ii) *$\alpha_{F(X)} = F(\alpha_X)$ for any $X \in \mathrm{Ob}(\mathcal{C})$.*

Lemma 2.1.4. *For two homogeneous symplectic manifolds $\mathfrak{X}$ and $\mathfrak{Y}$, we have*

$$\mathrm{K}_{\mathfrak{X}\times\mathfrak{Y}} \circ (\mathrm{K}_{\mathfrak{X}} \boxtimes \mathrm{K}_{\mathfrak{Y}}) \simeq \mathrm{K}_{\mathfrak{X}\times\mathfrak{Y}} \quad \textit{and} \quad \mathrm{K}_{\mathfrak{X}\times\mathfrak{Y}} \circ \mathrm{K}_{\mathfrak{X}} \simeq \mathrm{K}_{\mathfrak{X}\times\mathfrak{Y}}.$$

Proof. The last isomorphism is obtained by applying Proposition 1.3.12 to $(\mathfrak{X} \times \mathfrak{Y} \times \mathfrak{Y}, \mathfrak{X}, \mathfrak{X})$, and the first isomorphism follows from the second since

$$\mathrm{K}_{\mathfrak{X}\times\mathfrak{Y}} \circ (\mathrm{K}_{\mathfrak{X}} \boxtimes \mathrm{K}_{\mathfrak{Y}}) \simeq (\mathrm{K}_{\mathfrak{X}\times\mathfrak{Y}} \circ \mathrm{K}_{\mathfrak{X}}) \circ \mathrm{K}_{\mathfrak{Y}}\,.$$

□

Now let X be a manifold and set $\mathfrak{X} := T^*X$. Then $\mathfrak{X}$ has a canonical structure of a homogeneous symplectic manifold. The microlocalization functor is defined by:

$$\mu_X \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_{\mathfrak{X}})) \quad ; \quad \mathcal{F} \mapsto \mu_X \mathcal{F} := \mathrm{K}_{\mathfrak{X}} \circ \pi_X^{-1}\mathcal{F}.$$

The microlocalization functor μ_X may also be obtained as an integral transform associated with a kernel $\mathrm{L}_X \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_{T^*X\times X}))$ which is often easier to manipulate than $\mathrm{K}_{\mathfrak{X}}$.

Definition 2.1.5. *The kernel $\mathrm{L}_X \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_{\mathbb{K}_{T^*X\times X}}))$ is given by*

$$\mathrm{L}_X = \mathcal{L}_{\sigma_X}\big(T^*X \underset{X}{\times} X, T^*X \times X\big),$$

where σ_X is induced by ω_X on the first factor and $-id$ on the second factor.

Remark 2.1.6. Let $(x;\xi)$ be a local coordinate system on $\mathfrak{X} = T^*X$ and let $(x,\xi;\eta,y)$ denote the associated coordinates on $T^*\mathfrak{X}$. Then σ_X is defined by

$$\sigma_X(x;\xi) = ((x,\xi;\xi,0),(x;-\xi)) \in T^*\mathfrak{X} \times T^*X.$$

Therefore $\mathcal{T}(\sigma_X) = T^*X \underset{X}{\times} X$.

Proposition 2.1.7. *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$. There is a canonical isomorphism*

$$\mu_X \mathcal{F} \simeq L_X \circ \mathcal{F}.$$

Proof. Consider the following diagram

$$\begin{tikzcd}
& T^*X \times T^*X \arrow[d, "q"] \arrow[dr, "p_2"] \arrow[ddl, "p_1"'] & \\
& T^*X \times X \arrow[dl, "p_1'"] \arrow[dr, "p_2'"'] & T^*X \arrow[d, "\pi_X"] \\
T^*X & & X.
\end{tikzcd} \tag{2.2}$$

Since q satisfies the assumptions of Corollary 1.3.5, we have the isomorphism $\mathrm{R}q_{!!}\,\mathrm{K}_{\mathfrak{X}} \simeq \mathrm{L}_X$, which implies

$$\begin{aligned}
\mathrm{L}_X \circ \mathcal{F} \simeq \mathrm{R}p'_{1!!}\left(\mathrm{R}q_{!!}\,\mathrm{K}_{\mathfrak{X}} \otimes p'^{-1}_2 \mathcal{F}\right) &\simeq \mathrm{R}p'_{1!!}\,\mathrm{R}q_{!!}\left(\mathrm{K}_{\mathfrak{X}} \otimes q^{-1} p'^{-1}_2 \mathcal{F}\right) \\
&\simeq \mathrm{R}p_{1!!}\left(\mathrm{K}_{\mathfrak{X}} \otimes p_2^{-1} \pi_X^{-1} \mathcal{F}\right) \simeq \mathrm{K}_{\mathfrak{X}} \circ \pi_X^{-1} \mathcal{F} \simeq \mu_X \mathcal{F}.
\end{aligned}$$

$\square$

The next lemma immediately follows from Lemma 2.1.2.

Lemma 2.1.8. *For $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$, we have*

$$\mathrm{K}_{T^*X} \circ \mu_X \mathcal{F} \simeq \mu_X \mathcal{F}.$$

Example 2.1.9. Let $Z \subset X$ be a closed submanifold. Then

$$\mu_X(\mathbb{K}_Z) \simeq \mathcal{L}_{\omega_X}(T^*X \underset{X}{\times} Z, T^*X).$$

Indeed, noticing that $\mathbb{K}_Z \simeq \mathcal{L}_0(Z, X)$, it is enough to apply Proposition 1.3.12 to the triplet (T^*X, X, pt) with $N = T^*X \underset{X}{\times} Z$.

Note that the support of $\mu_X(\mathbb{K}_Z)$ is $T^*_Z X$. Let us take a local coordinate system (x, z) on X such that $Z = \{x = 0\}$. Let $(x, z; \xi, \zeta)$ be the corresponding coordinates on T^*X. Then on $\dot{T}^*X$, we have

$$\mu_X(\mathbb{K}_Z) \simeq \text{“}\varinjlim_{\varepsilon > 0}\text{”}\, \mathbb{K}_{\{-\langle \xi, x\rangle > \varepsilon |x|\}} \otimes \widetilde{\mathbb{K}}_{\{x=0,\, \zeta=0\}}[\operatorname{codim} Z].$$

Note that

$$\mu_X(\widetilde{\mathbb{K}}_Z) \simeq \widetilde{\mathbb{K}}_{T^*_X X \underset{X}{\times} Z}\,. \tag{2.3}$$

Lemma 2.1.10. *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}\left(\mathrm{I}\left(\mathbb{K}_{T^*X}\right)\right)$. Then*

$$(\mathrm{K}_{T^*X} \circ \mathcal{F}) \otimes \widetilde{\mathbb{K}}_{T^*_X X} \simeq \mathcal{F} \otimes \widetilde{\mathbb{K}}_{T^*_X X},$$

In particular if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}\left(\mathrm{I}\left(\mathbb{K}_X\right)\right)$ then

$$\mu_X \mathcal{F} \otimes \widetilde{\mathbb{K}}_{T^*_X X} \simeq \pi_X^{-1} \mathcal{F} \otimes \widetilde{\mathbb{K}}_{T^*_X X}\,.$$

Proof. With the notations in (2.2), we have an isomorphism by Proposition 1.2.8:

$$\mathrm{K}_{T^*X} \otimes p_1^{-1} \widetilde{\mathbb{K}}_{T^*_X X} \simeq \mathbb{K}_{\Delta_{T^*X}} \otimes p_1^{-1} \widetilde{\mathbb{K}}_{T^*_X X}\,.$$

Therefore we have for $\mathcal{F}\in\mathrm{D}^{\mathrm{b}}\left(\mathrm{I}\left(\mathbb{K}_{T^*X}\right)\right)$

$$\begin{aligned}\left(\mathrm{K}_{T^*X}\circ\mathcal{F}\right)\otimes\widetilde{\mathbb{K}}_{T^*_XX}&\simeq\mathrm{R}p_{1!!}\left(\mathrm{K}_{T^*X}\otimes p_2^{-1}\mathcal{F}\right)\otimes\widetilde{\mathbb{K}}_{T^*_XX}=\mathrm{R}p_{1!!}\Big(\mathrm{K}_{T^*X}\otimes p_1^{-1}\widetilde{\mathbb{K}}_{T^*_XX}\otimes p_2^{-1}\mathcal{F}\Big)\\&\simeq\mathrm{R}p_{1!!}\Big(\mathbb{K}_{\Delta_{T^*X}}\otimes p_1^{-1}\widetilde{\mathbb{K}}_{T^*_XX}\otimes p_2^{-1}\mathcal{F}\Big)=\mathrm{R}p_{1!!}\left(\mathbb{K}_{\Delta_{T^*X}}\otimes p_2^{-1}\mathcal{F}\right)\otimes\widetilde{\mathbb{K}}_{T^*_XX}\\&\simeq\mathcal{F}\otimes\widetilde{\mathbb{K}}_{T^*_XX}\,.\end{aligned}$$

$\square$

Remark 2.1.11. The ind-sheaf $\mu_X\mathcal{F}$ is conical in the sense that it is equivariant with respect to the $\mathbb{R}_{>0}$-action on T^*X. We will not develop here the theory of conic ind-sheaves but simply give some consequences sufficient for our purpose. Let $\dot{T}^*X$ be the cotangent bundle with its zero section removed, and S^*X the associated sphere bundle. Let $\gamma\colon\dot{T}^*X\to S^*X$ be the natural projection and $\mathcal{F}\in\mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$. Then we have the following isomorphism:

$$\mu_X\mathcal{F}|_{\dot{T}^*X}\simeq\gamma^{-1}\,\mathrm{R}\gamma_*\mu_X\mathcal{F}|_{\dot{T}^*X}.$$

Indeed, the kernel L_X satisfies a similar property.

Lemma 2.1.12. *Let X be a real manifold and $\pi_E\colon E\to X$ a real vector bundle over X. Denote by $\mathrm{S}E$ the spherical bundle associated with E and by*

$$j\colon\dot{E}\hookrightarrow E\qquad p\colon\dot{E}\to\mathrm{S}E$$

the natural morphisms. Assume that $\mathcal{F}\in\mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_E))$ satisfies $j^{-1}\mathcal{F}\simeq p^{-1}\mathcal{G}$ for some $\mathcal{G}\in\mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_{SE}))$. Then

(i) $\mathrm{R}\pi_{E*}\,\mathrm{R}j_{!!}j^{-1}\mathcal{F}\simeq 0$,

(ii) $\mathrm{R}\pi_{E*}(\mathcal{F})\xrightarrow{\sim}\mathrm{R}\pi_{E*}(\widetilde{\mathbb{K}}_X\otimes\mathcal{F})$*, where X is identified to the zero section of E,*

(iii) *there is a natural distinguished triangle*

$$\mathrm{R}\dot{\pi}_{E!!}j^{-1}\mathcal{F}\longrightarrow\mathrm{R}\pi_{E!!}\mathcal{F}\longrightarrow\mathrm{R}\pi_{E*}\mathcal{F}\xrightarrow{+1}.$$

Proof. (a) Let E_X denote the real blow up of E along X identified with the zero section, *i.e.* $E_X=\big(\dot{E}\times\mathbb{R}_{\geq 0}\big)/\mathbb{R}_{>0}$, hence $E_X=\dot{E}\sqcup SE$ as a set. We have the following commutative diagram

$$\begin{tikzcd}\dot{E}\arrow[r, hook, "i"]\arrow[dr, hook, "j"'] & E_X\arrow[r, "q"]\arrow[d, "\pi_{E_X}"] & \mathrm{S}E\arrow[d, "\pi_{\mathrm{S}E}"]\\ & E\arrow[r, "\pi_E"'] & X\end{tikzcd}$$

where π_{E_X} and π_{SE} are proper.

(b) We shall first show

$$\mathrm{R}q_*\,\mathrm{R}i_{!!}j^{-1}\mathcal{F}\simeq 0.$$

Since q is locally trivial with fiber $R_{\geq 0}$, we have $q^!\mathcal{G}\simeq q^{-1}\mathcal{G}\otimes q^!\,\mathbb{K}_{SE}\simeq q^{-1}\mathcal{G}\otimes\mathbb{K}_{i(\dot{E})}[1]$. Therefore we have

$$\mathrm{R}q_*(\mathbb{K}_{i(\dot{E})}\otimes q^{-1}\mathcal{G})\simeq\mathrm{R}q_*\,\mathrm{R}\mathcal{I}\mathcal{H}om\left(\mathbb{K}_{E_X}[1],q^!\mathcal{G}\right)\simeq\mathrm{R}\mathcal{I}\mathcal{H}om\left(\mathrm{R}q_{!!}\,\mathbb{K}_{E_X}[1],\mathcal{G}\right)\simeq 0$$

since $\mathrm{R}q_{!!}\,\mathbb{K}_{E_X}=0$. On the other hand, we have

$$\mathrm{R}q_*\left((\mathbb{K}_{i(\dot{E})}/\,\widetilde{\mathbb{K}}_{i(\dot{E})})\otimes q^{-1}\mathcal{G}\right)\simeq\mathrm{R}q_{!!}\left((\mathbb{K}_{i(\dot{E})}/\,\widetilde{\mathbb{K}}_{i(\dot{E})})\otimes q^{-1}\mathcal{G}\right)\simeq\mathrm{R}q_{!!}\left((\mathbb{K}_{i(\dot{E})}/\,\widetilde{\mathbb{K}}_{i(\dot{E})})\right)\otimes\mathcal{G}\simeq 0.$$

Hence the desired result follows from the distinguished triangle:

$$\mathrm{R}q_*(\widetilde{\mathbb{K}}_{i(\dot{E})}\otimes q^{-1}\mathcal{G})\longrightarrow\mathrm{R}q_*(\mathbb{K}_{i(\dot{E})}\otimes q^{-1}\mathcal{G})\longrightarrow\mathrm{R}q_*\left((\mathbb{K}_{i(\dot{E})}/\,\widetilde{\mathbb{K}}_{i(\dot{E})})\otimes q^{-1}\mathcal{G}\right)\xrightarrow{+1},$$

in which the first term is isomorphic to $\mathrm{R}q_* \,\mathrm{R}i_{!!} j^{-1}\mathcal{F}$.

(i) We have a chain of isomorphisms

$$\begin{aligned}\mathrm{R}\pi_{E*}\,\mathrm{R}j_{!!}j^{-1}\mathcal{F} &\simeq \mathrm{R}\pi_{E*}\,\mathrm{R}\pi_{E_X\,!!}\,\mathrm{R}i_{!!}j^{-1}\mathcal{F}\\ &\simeq \mathrm{R}\pi_{E*}\,\mathrm{R}\pi_{E_X*}\,\mathrm{R}i_{!!}j^{-1}\mathcal{F} \simeq \mathrm{R}\pi_{SE*}\,\mathrm{R}q_*\,\mathrm{R}i_{!!}j^{-1}\mathcal{F},\end{aligned}$$

which vanishes by (b).

(ii) Applying the functor $\mathrm{R}\pi_{E*}(\bullet \otimes \mathcal{F})$ to the distinguished triangle

$$\widetilde{\mathbb{K}}_{\dot{E}} \longrightarrow \mathbb{K}_E \longrightarrow \widetilde{\mathbb{K}}_X \xrightarrow{+1}, \tag{2.4}$$

we obtain the distinguished triangle

$$\mathrm{R}\pi_{E*}(\widetilde{\mathbb{K}}_{\dot{E}} \otimes \mathcal{F}) \longrightarrow \mathrm{R}\pi_{E*}\mathcal{F} \longrightarrow \mathrm{R}\pi_{E*}(\widetilde{\mathbb{K}}_X \otimes \mathcal{F}) \xrightarrow{+1},$$

in which the first term vanishes by (i).

(iii) Applying the functor $\mathrm{R}\pi_{E!!}(\bullet \otimes \mathcal{F})$ to the distinguished triangle (2.4), we obtain the distinguished triangle

$$\mathrm{R}\pi_{E!!}(\widetilde{\mathbb{K}}_{\dot{E}} \otimes \mathcal{F}) \longrightarrow \mathrm{R}\pi_{E!!}\mathcal{F} \longrightarrow \mathrm{R}\pi_{E!!}(\widetilde{\mathbb{K}}_X \otimes \mathcal{F}) \xrightarrow{+1},$$

in which the first term is isomorphic to $\mathrm{R}\dot{\pi}_{E!!}j^{-1}\mathcal{F}$ and the last term is isomorphic to $\mathrm{R}\pi_{E*}\mathcal{F}$ by (ii). □

Proposition 2.1.13. *Let* $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$. *Then*

(i) $\mathrm{R}\pi_{X*}\mu_X\mathcal{F} \simeq \mathcal{F}$,

(ii) $\mathrm{R}\pi_{X!!}\mu_X\mathcal{F} \simeq \widetilde{\mathbb{K}}_{\Delta_X} \circ \mathcal{F}$,

(iii) $\mathrm{R}\dot{\pi}_{X!!}(\mu_X\mathcal{F}|_{\dot{T}^*X}) \simeq \left(\mathbb{K}_{X\times X\setminus\Delta_X} \otimes \widetilde{\mathbb{K}}_{\Delta_X}\right) \circ \mathcal{F}$,

(iv) *there is a natural distinguished triangle*

$$\mathrm{R}\dot{\pi}_{X!!}(\mu_X\mathcal{F}|_{\dot{T}^*X}) \longrightarrow \mathrm{R}\pi_{X!!}\mu_X\mathcal{F} \longrightarrow \mathcal{F} \xrightarrow{+1} .$$

Proof. (i) By Lemma 2.1.12 (ii), we have

$$\mathrm{R}\pi_{X*}\mu_X\mathcal{F} \simeq \mathrm{R}\pi_{X*}\left(\mu_X\mathcal{F} \otimes \widetilde{\mathbb{K}}_{T^*_XX}\right) \simeq \mathrm{R}\pi_{X!!}\left(\pi_X^{-1}\mathcal{F} \otimes \widetilde{\mathbb{K}}_{T^*_XX}\right) \simeq \mathcal{F} \otimes \mathrm{R}\pi_{X!!}\,\widetilde{\mathbb{K}}_{T^*_XX} \simeq \mathcal{F},$$

where the second isomorphism follows from Lemma 2.1.10.

(ii) and (iii) Let us denote by $p\colon T^*X \times X \to X \times X$ the canonical morphism. Then we have isomorphisms:

$$\begin{aligned}\mathrm{R}\pi_{X!!}\,\mu_X\mathcal{F} &\simeq (\mathrm{R}p_{!!}L_X) \circ \mathcal{F},\\ \mathrm{R}\dot{\pi}_{X!!}(\mu_X\mathcal{F}|_{\dot{T}^*X}) &\simeq \left(\mathrm{R}p_{!!}(L_X \otimes \widetilde{\mathbb{K}}_{\dot{T}^*X\times X})\right) \circ \mathcal{F}.\end{aligned}$$

Hence, it is enough to show the isomorphism

$$\mathrm{R}p_{!!}L_X \simeq \widetilde{\mathbb{K}}_{\Delta_X}, \tag{2.5}$$

$$\mathrm{R}p_{!!}(L_X \otimes \widetilde{\mathbb{K}}_{\dot{T}^*X\times X}) \simeq \mathbb{K}_{X\times X\setminus\Delta_X} \otimes \widetilde{\mathbb{K}}_{\Delta_X}. \tag{2.6}$$

The natural morphism given in Corollary 1.2.10

$$\mathrm{L}_X \longrightarrow \widetilde{\mathbb{K}}_{T^*X\underset{X}{\times}X} \otimes \beta_{T^*X\times X}\left(\omega^{\otimes -1}_{T^*X\underset{X}{\times}X/T^*X\times X}\right) = p^!\,\widetilde{\mathbb{K}}_{\Delta_X}$$

provides a morphism $\mathrm{R}p_{!!}L_X \longrightarrow \widetilde{\mathbb{K}}_{\Delta_X}$.

We shall first show (2.6). Take a local coordinate system $x = (x_1, \dots, x_n)$ on X and let $((x;\xi), x')$ be the associated local coordinates on $T^*X \times X$. We have

$$\begin{aligned}
\mathrm{L}_X \otimes \widetilde{\mathbb{K}}_{\dot{T}^*X\times X} &\simeq \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{K}_{\left\{((x;\xi),x')\,;\,\langle \xi, x'-x\rangle > \varepsilon |x'-x|\right\}} \otimes \widetilde{\mathbb{K}}_{\dot{T}^*X\underset{X}{\times}X} \otimes \beta\left(\mathrm{R}i_* \omega^{\otimes -1}_{T^*X\underset{X}{\times}X/T^*X\times X}\right)\\
&\simeq \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{K}_{\left\{((x;\xi),x')\,;\,\langle \xi, x'-x\rangle > \varepsilon |x'-x|\right\}} \otimes p^{-1}\, \widetilde{\mathbb{K}}_{\Delta_X}[n].
\end{aligned}$$

Hence

$$\mathrm{R}p_{!!}\left(\mathrm{L}_X \otimes \widetilde{\mathbb{K}}_{\dot{T}^*X\times X}\right) \simeq \mathrm{R}p_{!!}\Big(\text{“}\varinjlim_{\varepsilon>0,\, R>0}\text{”}\ \mathbb{K}_{\left\{((x;\xi),x')\,;\,\langle \xi, x'-x\rangle > \varepsilon |x'-x|,\, |\xi|<R\right\}}\Big) \otimes \widetilde{\mathbb{K}}_{\Delta_X}[n].$$

For $0 < \varepsilon < R$, we have

$$\mathrm{R}^k p_!\Big(\mathbb{K}_{\left\{((x;\xi),x')\,;\,\langle \xi, x'-x\rangle > \varepsilon |x'-x|,\, |\xi|<R\right\}}\Big) \simeq \begin{cases} \mathbb{K}_{\{0<|x'-x|<\varepsilon^{-1}R\}} & \text{if } k = n.\\ 0 & \text{if } k \neq n.\end{cases}$$

Hence we have shown that

$$\mathrm{R}p_{!!}\left(\mathrm{L}_X \otimes \widetilde{\mathbb{K}}_{\dot{T}^*X\times X}\right) \ \simeq\ \mathbb{K}_{X\times X\setminus\Delta_X}[-n] \otimes \widetilde{\mathbb{K}}_{\Delta_X}[n] \simeq \mathbb{K}_{X\times X\setminus\Delta_X} \otimes \widetilde{\mathbb{K}}_{\Delta_X},$$

which proves (2.6). In the morphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}p_{!!}\left(\mathrm{L}_X \otimes \widetilde{\mathbb{K}}_{\dot{T}^*X\times X}\right) \arrow[r] \arrow[d, "\sim"] & \mathrm{R}p_{!!}\left(\mathrm{L}_X\right) \arrow[r] \arrow[d] & \mathrm{R}p_{!!}(\left(\mathrm{L}_X \otimes \widetilde{\mathbb{K}}_{T^*_XX\times X}\right) \arrow[r, "+1"] \arrow[d] & {} \\
\mathbb{K}_{X\times X\setminus\Delta_X} \otimes \widetilde{\mathbb{K}}_{\Delta_X} \arrow[r] & \widetilde{\mathbb{K}}_{\Delta_X} \arrow[r] & \mathbb{K}_{\Delta_X} \arrow[r, "+1"] & {},
\end{tikzcd}$$

the left vertical arrow is an isomorphism by (2.6) and the right vertical arrow is an isomorphism since

$$\mathrm{R}p_{!!}\left(\mathrm{L}_X \otimes \widetilde{\mathbb{K}}_{T^*_XX\times X}\right) \simeq \mathrm{R}p_{!!}\left(\mathbb{K}_{T^*X\underset{X}{\times}X} \otimes \widetilde{\mathbb{K}}_{T^*_XX\times X}\right) \simeq \mathbb{K}_{\Delta_X} \otimes \mathrm{R}p_{!!}(\widetilde{\mathbb{K}}_{T^*_XX\times X}) \simeq \mathbb{K}_{\Delta_X}\,.$$

Hence we obtain (2.5).

(iv) follows immediately from Lemma 2.1.12 and (i). □

Proposition 2.1.14. *For $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$ and $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_Y))$, we have an isomorphism*

$$\mu_{X\times Y}(\mathcal{F} \boxtimes \mathcal{G}) \simeq \mathrm{K}_{T^*(X\times Y)} \circ (\mu_X\mathcal{F} \boxtimes \mu_Y\mathcal{G})\,.$$

Proof. This follows immediately from Lemma 2.1.4. □

2.2. The link with μhom and classical microlocalization.

Proposition 2.2.1. *Let $\sigma \in \Gamma(X, \Omega^1_X)$ and $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$. Then we have an isomorphism*

$$\sigma^{-1}\,\mu hom(\mathcal{F}, \mathcal{G}) \simeq \mathrm{R}\mathcal{H}om\left(\mathcal{F}, \mathcal{L}_{\widetilde{\sigma}}(\Delta_X, X\times X) \circ \mathcal{G}\right),$$

where $\widetilde{\sigma} = q_1^*\sigma - q_2^*\sigma$ and $q_i\colon X\times X \to X$ is the i-th projection $(i = 1, 2)$.

Proof. By definition we have

$$\mu hom(\mathcal{F}, \mathcal{G}) \simeq \nu hom(\mathcal{F}, \mathcal{G})^\wedge,$$

where νhom is the specialization of the functor $\mathrm{R}\mathcal{H}om$ (see below), and $(\cdot)^\wedge$ is the Fourier-Sato transform. Setting

$$P' = \left\{((x;\xi),(x;v)) \in T^*X \underset{X}{\times} TX; \langle \xi, v\rangle \leqslant 0\right\},$$

the Fourier-Sato transform is the integral transform with kernel $\mathbb{K}_{P'}$. Consider the following commutative diagram

$$\begin{tikzcd}
TX & \\
T^*X \underset{X}{\times} TX \arrow[u, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & TX \arrow[l, "\sigma'"] \arrow[ul, "\mathrm{id}_{TX}"'] \arrow[d, "\tau_X"] \\
T^*X & X. \arrow[l, "\sigma"]
\end{tikzcd}$$

Then $\mu hom(\mathcal{F}, \mathcal{G}) \simeq \nu hom(\mathcal{F}, \mathcal{G})^\wedge \simeq \mathrm{R}\pi_{1!}\left(\pi_2^{-1}\, \nu hom(\mathcal{F}, \mathcal{G}) \otimes \mathbb{K}_{P'}\right)$. Hence

$$\begin{aligned}
\sigma^{-1}\,\mu hom(\mathcal{F}, \mathcal{G}) &\simeq \sigma^{-1}\,\mathrm{R}\pi_{1!}\left(\pi_2^{-1}\, \nu hom(\mathcal{F}, \mathcal{G}) \otimes \mathbb{K}_{P'}\right) \\
&\simeq \mathrm{R}\tau_{X!}\sigma'^{-1}\left(\pi_2^{-1}\, \nu hom(\mathcal{F}, \mathcal{G}) \otimes \mathbb{K}_{P'}\right) \simeq \mathrm{R}\tau_{X!}\left(\nu hom(\mathcal{F}, \mathcal{G}) \otimes \mathbb{K}_{P'_\sigma}\right),
\end{aligned}$$

where we have set $P'_\sigma = \sigma'^{-1}(P') = \{(x,v) \in TX; \langle \sigma(x), v\rangle \leqslant 0\}$. Consider the normal deformation of Δ_X in $X \times X$, visualized by the diagram:

$$\begin{tikzcd}
TX \arrow[r, "\sim"] & T_{\Delta_X}(X \times X) \arrow[r, hook, "s"] \arrow[d, "\tau_X"] & \widetilde{X \times X} \arrow[d, "p"] \arrow[dd, bend right, "p_1"'] \arrow[dd, bend left, "p_2"] & \Omega \arrow[l, hook', "j"'] \arrow[dl, "\widetilde{p}"] \\
& X \arrow[r, hook, "i"'] \arrow[dr, "\mathrm{id}_X"'] & X \times X \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] & \\
& & X &
\end{tikzcd}$$

Then $\nu hom(\mathcal{F}, \mathcal{G})$ is by definition $s^{-1}\,\mathrm{R}j_*\widetilde{p}^{-1}\,\mathrm{R}\mathcal{H}om(q_2^{-1}\mathcal{F}, q_1^!\mathcal{G})$. Since $\widetilde{p}$ is smooth we have

$$\widetilde{p}^{-1}\,\mathrm{R}\mathcal{H}om(q_2^{-1}\mathcal{F}, q_1^!\mathcal{G}) \simeq \mathrm{R}\mathcal{H}om(\widetilde{p}^{-1}q_2^{-1}\mathcal{F}, \widetilde{p}^{-1}q_1^!\mathcal{G}) \simeq \mathrm{R}\mathcal{H}om(j^{-1}p_2^{-1}\mathcal{F}, j^{-1}p^{-1}q_1^!\mathcal{G}).$$

Hence we have

$$\begin{aligned}
\nu hom(\mathcal{F}, \mathcal{G}) &\simeq s^{-1}\,\mathrm{R}j_*\,\mathrm{R}\mathcal{H}om(j^{-1}p_2^{-1}\mathcal{F}, j^{-1}p^{-1}q_1^!\mathcal{G}) \simeq s^{-1}\,\mathrm{R}\mathcal{H}om(p_2^{-1}\mathcal{F}, \mathrm{R}j_*j^{-1}p^{-1}q_1^!\mathcal{G}) \\
&\simeq s^{-1}\,\mathrm{R}\mathcal{H}om(p_2^{-1}\mathcal{F}, \mathrm{R}j_*j^{-1}p_1^{-1}\mathcal{G}) \otimes \tau_X^{-1}\omega_X.
\end{aligned}$$

Since p_1 is smooth, we have the estimate

$$\mathrm{SS}(p_1^{-1}\mathcal{G}) \cap \mathrm{SS}\,\mathbb{K}_\Omega \subset (p_1)_\pi^{-1}(T^*X) \cap \left(T^*_{T_{\Delta_X}(X\times X)}\widetilde{X \times X} \cup T^*_{\widetilde{X\times X}}\widetilde{X \times X}\right) \subset T^*_{\widetilde{X\times X}}\widetilde{X \times X},$$

which implies

$$\begin{aligned}
\mathrm{R}j_*j^{-1}p_1^{-1}\mathcal{G} &\simeq \mathrm{R}\mathcal{H}om(\mathbb{K}_\Omega, p_1^{-1}\mathcal{G}) \simeq \mathrm{R}\mathcal{H}om(\mathbb{K}_\Omega, \mathbb{K}_X) \otimes p_1^{-1}\mathcal{G} \\
&\simeq \mathbb{K}_{\overline{\Omega}} \otimes p_1^{-1}\mathcal{G}.
\end{aligned}$$

Applying this result we obtain

$$\begin{aligned}\nu hom(\mathcal{F},\mathcal{G}) &\simeq s^{-1}\,\mathrm{R}\mathcal{H}om(p_2^{-1}\mathcal{F}, p_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}})\otimes\tau_X^{-1}\omega_X\\ &\simeq s^{-1}\,\mathrm{R}\mathcal{H}om(p_2^{-1}\mathcal{F}, p_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}})\otimes\tau_X^{-1}\omega_{\Delta_X/X\times X}^{\otimes -1},\end{aligned}$$

and finally

$$\begin{aligned}\sigma^{-1}\,\mu hom(\mathcal{F},\mathcal{G}) &\simeq \mathrm{R}\tau_{X!}\Big(s^{-1}\,\mathrm{R}\mathcal{H}om\,(p_2^{-1}\mathcal{F}, p_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}})\otimes\tau_X^{-1}\omega_{\Delta_X/X\times X}^{\otimes -1}\otimes\mathbb{K}_{P'_\sigma}\Big)\\ &\simeq \mathrm{R}p_{2!}\,\mathrm{R}s_!\Big(s^{-1}\,\mathrm{R}\mathcal{H}om\,(p_2^{-1}\mathcal{F}, p_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}})\otimes\tau_X^{-1}\omega_{\Delta_X/X\times X}^{\otimes -1}\otimes\mathbb{K}_{P'_\sigma}\Big)\\ &\simeq \mathrm{R}p_{2!}\Big(\mathrm{R}\mathcal{H}om\,(p_2^{-1}\mathcal{F}, p_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}})\otimes\mathbb{K}_{P'_\sigma}\otimes p^{-1}\,\mathrm{R}i_*\omega_{\Delta_X/X\times X}^{\otimes -1})\Big).\end{aligned}$$

Note that this intermediate result is obtained by means of classical sheaf theory. However, formulas in the derived category of ind-sheaves allow us to continue the calculations. Using the properties (1.3 c) of the functor β and Proposition 1.1.2, we have

$$\begin{aligned}\sigma^{-1}\,\mu hom(\mathcal{F},\mathcal{G}) &\simeq \mathrm{R}p_{2!}\,\mathrm{R}\mathcal{H}om\Big(p_2^{-1}\mathcal{F}, p_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}}\otimes\beta_{\widetilde{X\times X}}(\mathbb{K}_{P'_\sigma}\otimes p^{-1}\,\mathrm{R}i_*\omega_{\Delta_X/X\times X}^{\otimes -1})\Big)\\ &\simeq \mathrm{R}\mathcal{H}om\Big(\mathcal{F}, \mathrm{R}p_{2!!}\big(p_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}}\otimes\widetilde{\mathbb{K}}_{P'_\sigma}\otimes p^{-1}\beta_{X\times X}(\mathrm{R}i_*\omega_{\Delta_X/X\times X}^{\otimes -1})\big)\Big).\end{aligned}$$

We have furthermore

$$\begin{aligned}&\mathrm{R}p_{2!!}\big(p_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}}\otimes\widetilde{\mathbb{K}}_{P'_\sigma}\otimes p^{-1}\beta_{X\times X}(\mathrm{R}i_*\omega_{\Delta_X/X\times X}^{\otimes -1})\big)\\ &\qquad\simeq \mathrm{R}q_{2!!}\,\mathrm{R}p_{!!}\Big(p^{-1}q_1^{-1}\mathcal{G}\otimes\mathbb{K}_{\overline{\Omega}}\otimes\widetilde{\mathbb{K}}_{P'_\sigma}\otimes p^{-1}\beta_{X\times X}(\mathrm{R}i_*\omega_{\Delta_X/X\times X}^{\otimes -1})\Big)\\ &\qquad\simeq \mathrm{R}q_{2!!}\Big(q_1^{-1}\mathcal{G}\otimes\mathrm{R}p_{!!}(\mathbb{K}_{\overline{\Omega}}\otimes\widetilde{\mathbb{K}}_{P'_\sigma})\otimes\beta_{X\times X}(\mathrm{R}i_*\omega_{\Delta_X/X\times X}^{\otimes -1})\Big)\\ &\qquad\simeq \mathrm{R}q_{1!!}\Big(q_2^{-1}\mathcal{G}\otimes\mathrm{R}p_{!!}(\mathbb{K}_{\overline{\Omega}}\otimes\widetilde{\mathbb{K}}_{P_{\tilde\sigma}})\otimes\beta_{X\times X}(\mathrm{R}i_*\omega_{\Delta_X/X\times X}^{\otimes -1})\Big)\\ &\qquad\simeq \mathrm{R}q_{1!!}\big(q_2^{-1}\mathcal{G}\otimes\mathcal{L}_{\tilde\sigma}(\Delta_X, X\times X)\big)\simeq\mathcal{L}_{\tilde\sigma}(\Delta_X, X\times X)\circ\mathcal{G}.\end{aligned}$$

□

Corollary 2.2.2. *Let $\mathcal{F},\mathcal{G}\in\mathrm{D^b}(\mathbb{K}_X)$. Then we have an isomorphism*

$$\mu hom(\mathcal{F},\mathcal{G})\simeq\mathrm{R}\mathcal{H}om(\pi_X^{-1}\mathcal{F},\mu_X\mathcal{G})\simeq\mathrm{R}\mathcal{H}om(\mu_X\mathcal{F},\mu_X\mathcal{G}).$$

Proof. Consider the fundamental 1-form $\omega_X\in\Gamma(T^*X,\Omega^1_{T^*X})$ of the cotangent bundle of X. Then we have

$$\mu hom(\mathcal{F},\mathcal{G})\simeq\omega_X^{-1}\,\mu hom(\pi_X^{-1}\mathcal{F},\pi_X^{-1}\mathcal{G})$$

and by Proposition 2.2.1 we get a natural isomorphism

$$\mu hom(\mathcal{F},\mathcal{G})=\mathrm{R}\mathcal{H}om(\pi_X^{-1}\mathcal{F},\mathrm{K}_{T^*X}\circ\pi_X^{-1}\mathcal{G})\simeq\mathrm{R}\mathcal{H}om(\pi_X^{-1}\mathcal{F},\mu_X\mathcal{G})$$

The last isomorphism is a consequence of Lemma 2.1.3 and Lemma 2.1.2. □

Proposition 2.2.3. *Let $\mathcal{F}\in\mathrm{D^b}(\mathbb{K}_X)$ and let Z be a closed submanifold of X. Denote by i the closed immersion $i\colon T^*X\underset{X}{\times}Z\hookrightarrow T^*X$. Then we have a natural isomorphism*

$$\mu_Z(\mathcal{F})\simeq\alpha_{T^*X\underset{X}{\times}Z}(i^!\mu_X\mathcal{F})|_{T^*_ZX})\simeq\mathrm{R}\mathcal{H}om(\mathbb{K}_{T^*X\underset{X}{\times}Z},\mu_X\mathcal{F})|_{T^*_ZX}.$$

Here $\mu_Z(\mathcal{F})$ denotes the classical functor of Sato's microlocalization

See [KS2], Chapter IV for definitions and a detailed study for μ_Z. We only remark here that $\mu_Z(\mathcal{F})\simeq\mu hom(\mathbb{K}_Z,\mathcal{F})|_{T^*_ZX}$.

Proof. We have by Corollary 2.2.2

$$\begin{aligned}\mu_Z(\mathcal{F}) &\simeq \mathrm{R}\mathcal{H}om(\pi_X^{-1}\mathbb{K}_Z, \mu_X(\mathcal{F}))|_{T^*_Z X} \simeq \mathrm{R}\mathcal{H}om\big(\mathrm{R}i_{!!}\,\mathbb{K}_{T^*X\underset{X}{\times}Z}, \mu_X(\mathcal{F})\big)|_{T^*_Z X}\\ &\simeq \mathrm{R}\mathcal{H}om\big(\mathbb{K}_{T^*X\underset{X}{\times}Z}, i^!\mu_X(\mathcal{F})\big)|_{T^*_Z X} \simeq \big(\alpha_{T^*X\underset{X}{\times}Z}\, i^!\mu_X(\mathcal{F})\big)|_{T^*_Z X}.\end{aligned}$$

□

2.3. **Review on the microsupport of ind-sheaves.** In this section we shall give a short overview on the results of [KS4] on the microsupport of ind-sheaves .

The microsupport $\mathrm{SS}(\mathcal{F})$ of an object $\mathcal{F} \in \mathrm{D^b}(\mathbb{K}_X)$ is a closed involutive cone in the cotangent bundle T^*X which describes the codirections in which the cohomology of $\mathcal{F}$ does not propagate (cf. [KS2], [KS3]). The corresponding notions for ind-sheaves are more intricate.

Let $\mathcal{C}$ be an abelian category, and consider the functor

$$\mathrm{J}\colon \mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \longrightarrow \mathrm{D^b}(\mathcal{C})^\wedge \qquad \text{given by} \qquad \mathcal{F} \mapsto \mathrm{Hom}_{\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))}(\,\cdot\,, \mathcal{F}).$$

Here, $\mathrm{D^b}(\mathcal{C})^\wedge$ is the category of contravariant functors from $\mathrm{D^b}(\mathcal{C})$ to the category of sets. Then it can be shown that J factors through $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$. Note that J is conservative, which is a consequence of the commutative diagram

$$\begin{tikzcd} \mathrm{D^b}(\mathrm{Ind}(\mathcal{C})) \arrow[rr, "\mathrm{J}"] \arrow[dr, "\mathrm{H}^k"'] & & \mathrm{Ind}(\mathrm{D^b}(\mathcal{C})) \arrow[dl, "\mathrm{IH}^k"] \\ & \mathrm{Ind}(\mathcal{C}) & \end{tikzcd}$$

Finally assume

(2.7) $\qquad$ $\mathcal{C}$ has enough injectives and finite homological dimension.

Recall that in this case $\varphi\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism in $\mathrm{Ind}(\mathrm{D^b}(\mathcal{C}))$ if and only if $\mathrm{IH}^k(\varphi)$ is an isomorphism for all k. Then we easily get the following result.

Lemma 2.3.1. *Assume* (2.7)*. Let* $\mathcal{F} \in \mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$ *and let* $\{\mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ *be a filtrant inductive system of morphisms in* $\mathrm{D^b}(\mathrm{Ind}(\mathcal{C}))$*. Then* "$\varinjlim$" $\mathrm{J}(\mathcal{F}_i) \xrightarrow{\sim} \mathrm{J}(\mathcal{F})$ *if and only if* "$\varinjlim\limits_{i\in I}$" $\mathrm{H}^k(\mathcal{F}_i) \xrightarrow{\sim} \mathrm{H}^k(\mathcal{F})$.

In particular if "$\varinjlim$" $\mathrm{J}(\mathcal{F}_i) \xrightarrow{\sim} \mathrm{J}(\mathcal{F})$ the we have "$\varinjlim$" $\mathrm{J}(\tau^{\leqslant n}\mathcal{F}_i) \xrightarrow{\sim} \mathrm{J}(\tau^{\leqslant n}\mathcal{F})$ for all k.

We shall apply the results above to the case of ind-sheaves, by taking $\mathrm{Mod^c}(\mathbb{K}_X)$ as $\mathcal{C}$. For a C^∞-manifold X, let

$$J_X\colon \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X)) \to \big(\mathrm{D^b}(\mathrm{Mod^c}(\mathbb{K}_X))\big)^\wedge$$

be the canonical functor.

Proposition 2.3.2. *Let* $f\colon X \to Y$ *be a continuous map. Let* $\{\mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ *be a filtrant inductive system of morphisms in* $\mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$ *and* $\{\mathcal{G}_j \to \mathcal{G}\}_{j\in J}$ *a filtrant inductive system in* $\mathrm{D^b}(\mathrm{I}(\mathbb{K}_Y))$ *such that*

$$J_X(\mathcal{F}) \simeq \text{“}\varinjlim_{i\in I}\text{”}\, J_X(\mathcal{F}_i) \qquad \textit{and} \qquad J_Y(\mathcal{G}) \simeq \text{“}\varinjlim_{j\in J}\text{”}\, J_Y(\mathcal{G}_j).$$

Then

(i)
$$J_Y(\mathrm{R}f_{!!}\mathcal{F}) \simeq \text{“}\varinjlim_{i\in I}\text{”}\, J_Y(\mathrm{R}f_{!!}\mathcal{F}_i),$$

(ii) *For* $\mathcal{K} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$*, we have*
$$J_X(\mathcal{K}\otimes\mathcal{F}) = \text{“}\varinjlim_{i\in I}\text{”}\, J_X(\mathcal{K}\otimes\mathcal{F}_i),$$

(iii)
$$J_X(f^{-1}\mathcal{G}) \simeq \text{“}\varinjlim_{j\in J}\text{”}\, J_X(f^{-1}\mathcal{G}_j) \quad \textit{and} \quad J_X(f^{!}\mathcal{G}) \simeq \text{“}\varinjlim_{j\in J}\text{”}\, J_X(f^{!}\mathcal{G}_j)$$

(iv)
$$J_{T^*X}(\mu_X\mathcal{F}) \simeq \text{“}\varinjlim_{i\in I}\text{”}\, J_{T^*X}(\mu_X\mathcal{F}_i).$$

Proof. By Lemma 2.3.1, we can reduce the situation by dévissage to usual ind-sheaves, where the formulas are obvious. □

Definition 2.3.3. (i) *Let* $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$*. The micro-support of* $\mathcal{F}$*, denoted* $\mathrm{SS}(\mathcal{F})$*, is the closed conic subset of* T^*X *whose complementary is the set of points* $p \in T^*X$ *such that there exist a conic open neighborhood* U *of* p *in* T^*X*, an open neighborhood* W *of* $\pi_X(p)$ *and a small filtrant inductive system* $\{\mathcal{F}_i\}_{i\in I}$ *of objects* $\mathcal{F}_i \in \mathrm{D^b}(\mathrm{Mod^c}(\mathbb{K}_X))$ *such that* $\mathrm{SS}(\mathcal{F}_i)\cap U = \varnothing$ *and*
$$J_X(\mathcal{F}\otimes\mathbb{K}_W) \simeq \text{“}\varinjlim_{i\in I}\text{”}\, \mathcal{F}_i\otimes\mathbb{K}_W.$$

(ii) *For* $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$*, one sets* $\mathrm{SS}_0(\mathcal{F}) = \mathrm{Supp}(\mu_X\mathcal{F})$*.*

Remark 2.3.4. The micro-support defined above coincide with the classical definition for objects of $\mathrm{D^b}(\mathbb{K}_X)$, it satisfies the triangular inequality (in a distinguished triangle, the micro-support of an object is contained in the union of the micro-supports of the two others), and we have
$$\mathrm{Supp}(\mathcal{F}) = \mathrm{SS}(\mathcal{F})\cap T^*_XX, \quad \mathrm{SS}(\alpha_X(\mathcal{F}))\subset\mathrm{SS}(\mathcal{F}) \quad \text{for } \mathcal{F}\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_X)).$$
In general, it is no longer an involutive subset of T^*X.

Proposition 2.3.5. *Let* $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$*. Then*
$$\mathrm{SS}_0(\mathcal{F}) \subset \mathrm{SS}(\mathcal{F}).$$
If $\mathcal{F} \in \mathrm{D^b}(\mathbb{K}_X)$*, then*
$$\mathrm{SS}_0(\mathcal{F}) = \mathrm{SS}(\mathcal{F}).$$

Proof. The result for sheaves is actually an obvious consequence of Corollary 2.2.2 since
$$\mathrm{SS}(\mathcal{F}) = \mathrm{Supp}(\mu hom(\mathcal{F},\mathcal{F})) = \mathrm{Supp}(\mathrm{R}\mathcal{H}om(\mu_X\mathcal{F},\mu_X\mathcal{F})) = \mathrm{Supp}(\mu_X\mathcal{F}).$$
Now assume that $\mathcal{F}\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$ and $p\notin\mathrm{SS}\,\mathcal{F}$. Consider a filtrant inductive system $\mathcal{F}_i$ in $\mathrm{D^b}(\mathrm{Mod^c}(\mathbb{K}_X))$ and an open neighborhood W of $\pi_X(p)$, a neighborhood $U\subset{\pi_{T^*X}}^{-1}(W)$ of p such that
$$J_X(\mathcal{F}\otimes\mathbb{K}_W) \simeq \text{“}\varinjlim_{i}\text{”}(\mathcal{F}_i\otimes\mathbb{K}_W)$$
and $\mathrm{SS}(\mathcal{F}_i)\cap\overline{U} = \varnothing$. We have by Proposition 2.3.2
$$J_X\big(\mu_X(\mathcal{F}\otimes\mathbb{K}_W)\big) \simeq \text{“}\varinjlim_{i}\text{”}\, J_X\big(\mu_X(\mathcal{F}_i\otimes\mathbb{K}_W)\big),$$

and we get $\mu_X\mathcal{F}|_U \simeq 0$ since $\mathrm{Supp}(\mu_X\mathcal{F}_i) = \mathrm{SS}\,(F_i)$. $\square$

Example 2.3.6. For a closed submanifold Z of X, we have

$$\mathrm{SS}_0(\mathbb{K}_Z) = \mathrm{SS}(\mathbb{K}_Z) = T^*_Z X \quad \text{and}$$
$$\mathrm{SS}_0(\widetilde{\mathbb{K}}_Z) = T^*_X X \underset{X}{\times} Z, \quad \mathrm{SS}(\widetilde{\mathbb{K}}_Z) = T^*_Z X.$$

Lemma 2.3.7. *Let Ω be an open subset of $\dot{T}^*X$ and let $\mathcal{F} \in \mathrm{D^b}(\mathbb{K}_\Omega)$, $\mathcal{G} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_\Omega))$. Assume that $\mathcal{F}$ is cohomologically constructible* (*see* [KS2, Definition 3.4.1]). *Assume further*

$$\omega_X^{-1}\big(\mathrm{SS}(\mathcal{F})\big) \cap \mathrm{Supp}(\mathcal{G}) = \varnothing,$$

where ω_X is considered as a map $T^*X \to T^*(T^*X)$. Then we have an isomorphism

$$\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathbb{K}_\Omega) \otimes (\mathrm{K}_\Omega \circ \mathcal{G}) \xrightarrow{\sim} \mathrm{R}\mathcal{I}\mathcal{H}om(\mathcal{F}, \mathrm{K}_\Omega \circ \mathcal{G}) \quad \textit{in } \mathrm{D^b}(\mathrm{I}(\mathbb{K}_\Omega)).$$

Proof. By shrinking Ω, we may assume from the beginning that $\omega_X^{-1}\big(\mathrm{SS}(\mathcal{F})\big) = \varnothing$.

(i) Assume first that $\mathcal{G} \in \mathrm{D^b}(\mathbb{K}_\Omega)$. For $p = (x_0, \xi_0) \in \Omega$, we shall prove that

$$\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathbb{K}_\Omega) \otimes (\mathrm{K}_\Omega \circ \mathcal{G}) \otimes \widetilde{\mathbb{K}}_p \xrightarrow{\sim} \mathrm{R}\mathcal{I}\mathcal{H}om(\mathcal{F}, \mathrm{K}_\Omega \circ \mathcal{G}) \otimes \widetilde{\mathbb{K}}_p\,.$$

Since $p \notin T^*_X X$, we have:

$$(\mathrm{K}_\Omega \circ \mathcal{G}) \otimes \widetilde{\mathbb{K}}_p \;\simeq\; (\mathrm{K}_\Omega \otimes \widetilde{\mathbb{K}}_{(p,p)}) \circ \mathcal{G} \simeq \text{“}\varinjlim_{\rho>0}\text{”}\, \mathbb{K}_{K_\rho} \otimes\Big(\big(\text{“}\varinjlim_{\delta>0,\varepsilon>0}\text{”}\; \mathbb{K}_{F_{\delta,\varepsilon}}\big)[-n] \circ \mathcal{G}\Big), \tag{2.8}$$

where

$$K_\rho = \{(x, \xi); |x - x_0| \le \rho,\ |\xi - \xi_0| \le \rho\}$$

and

$$F_{\delta,\varepsilon} = \big\{\delta \geqslant \langle \xi_0, x' - x\rangle > \varepsilon(|x' - x| + |\xi' - \xi|)\big\}.$$

Let $p_1 \colon T^*\Omega \times T^*\Omega \to T^*\Omega$ be the first projection. For sufficiently small ε, δ and ρ, $\pi_{\mathfrak{X}}^{-1}K_\rho \cap p_1\big(\mathrm{SS}(\mathbb{K}_{F_{\delta,\varepsilon}})\big)$ is contained in a sufficiently small neighborhood of $\omega_X(p)$, and hence so is $\pi_{\mathfrak{X}}^{-1}K_\rho \cap \mathrm{SS}(\mathbb{K}_{F_{\delta,\varepsilon}} \circ \mathcal{G})$. Thus we obtain by assumption

$$\pi_{\mathfrak{X}}^{-1}K_\rho \cap \mathrm{SS}\,\mathcal{F} \cap \mathrm{SS}(\mathbb{K}_{F_{\delta,\varepsilon}} \circ \mathcal{G}) \subset T^*_{\mathfrak{X}}\mathfrak{X}.$$

Then by [KS2, Corollary 6.4.3], we have an isomorphism

$$\mathbb{K}_{K_\rho} \otimes \mathrm{R}\mathcal{H}om(\mathcal{F}, \mathbb{K}_\Omega) \otimes (\mathbb{K}_{F_{\delta,\varepsilon}} \circ \mathcal{G}) \xrightarrow{\sim} \mathbb{K}_{K_\rho} \otimes \mathrm{R}\mathcal{H}om(\mathcal{F}, \mathbb{K}_{F_{\delta,\varepsilon}} \circ \mathcal{G})$$

in $\mathrm{D^b}(\mathbb{K}_\Omega)$. Therefore we have

$$\begin{aligned}
&J_\Omega\,\big(\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathbb{K}_\Omega) \otimes (\mathrm{K}_\Omega \circ \mathcal{G}) \otimes \widetilde{\mathbb{K}}_p\big)\\
&\qquad \simeq \text{“}\varinjlim_{\delta>0,\,\varepsilon>0,\,\rho>0}\text{”}\; J_\Omega\,\big(\mathbb{K}_{K_\rho} \otimes \mathrm{R}\mathcal{H}om(\mathcal{F}, \mathbb{K}_\Omega) \otimes (\mathbb{K}_{F_{\delta,\varepsilon}} \circ \mathcal{G})[-n]\big)\\
&\qquad \simeq \text{“}\varinjlim_{\delta>0,\,\varepsilon>0,\,\rho>0}\text{”}\; J_\Omega\Big(\mathbb{K}_{K_\rho} \otimes \mathrm{R}\mathcal{H}om\big(\mathcal{F}, \mathbb{K}_{F_{\delta,\varepsilon}} \circ \mathcal{G}[-n]\big)\Big)\\
&\simeq J_\Omega\,\Big(\mathrm{R}\mathcal{I}\mathcal{H}om(\mathcal{F}, \mathrm{K}_\Omega \circ \mathcal{G}) \otimes \widetilde{\mathbb{K}}_p\Big),
\end{aligned}$$

and the lemma is proved when $\mathcal{G} \in \mathrm{D^b}(\mathbb{K}_\Omega)$.

In the general case, taking a filtrant inductive system $\mathcal{G}_k$ in $\mathrm{D}^{\mathrm{b}}(\mathbb{K}_\Omega)$ such that $J_\Omega(\mathcal{G}) \simeq$ "$\varinjlim$" $\mathcal{G}_k$. we have

$$\begin{aligned} J_\Omega\left(\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_\Omega) \otimes(\mathrm{K}_\Omega \circ \mathcal{G})\right) &\simeq \text{“}\varinjlim_k\text{”}\, J_\Omega\left(\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_\Omega) \otimes (\mathrm{K}_\Omega \circ \mathcal{G}_k)\right) \\ &\simeq \text{“}\varinjlim_k\text{”}\, J_\Omega\left(\mathrm{R}\mathcal{H}om(\mathcal{F}, K_\Omega \circ \mathcal{G}_k)\right) \simeq J_\Omega\Big(\mathrm{R}\mathcal{H}om(\mathcal{F},\mathrm{K}_\Omega \circ \mathcal{G})\Big), \end{aligned}$$

which completes the proof. $\square$

We prove now in the framework of ind-sheaves a well known result for sheaves.

Proposition 2.3.8. *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$ and $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$. Assume that $\mathcal{F}$ is cohomologically constructible. Assume further the non-characteristic condition*

$$\mathrm{SS}(\mathcal{F}) \cap \mathrm{SS}_0(\mathcal{G}) \subset T^*_X X.$$

Then we have an isomorphism

$$\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_X) \otimes \mathcal{G} \xrightarrow{\sim} \mathrm{R}\mathcal{I}\mathcal{H}om(\mathcal{F},\mathcal{G}).$$

Proof. Since $\omega_X^{-1}\,\mathrm{SS}(\pi_X^{-1}\mathcal{F}) = \mathrm{SS}\,\mathcal{F}$, the non-characteristic condition may be rewritten as

$$\omega_X^{-1}\,\mathrm{SS}(\pi_X^{-1}\mathcal{F}) \cap \operatorname{Supp}\mu_X\mathcal{G} \cap \dot{T}^*X = \varnothing,$$

and Lemma 2.3.7 assures that

$$\begin{aligned} \Big(\pi_X^{-1}\,\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_X) \otimes \mu_X\mathcal{G}\Big)|_{\dot{T}^*X} &\simeq \Big(\mathrm{R}\mathcal{H}om(\pi_X^{-1}\mathcal{F},\mathbb{K}_{T^*X}) \otimes \mu_X\mathcal{G}\Big)|_{\dot{T}^*X} \\ &\simeq \mathrm{R}\mathcal{I}\mathcal{H}om(\pi_X^{-1}\mathcal{F},\mu_X\mathcal{G})|_{\dot{T}^*X}. \end{aligned}$$

Applying the functor $\mathrm{R}\dot{\pi}_{X!!}$, we obtain

$$\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_X) \otimes \mathrm{R}\dot{\pi}_{X!!}\left(\mu_X\mathcal{G}|_{\dot{T}^*X}\right) \simeq \mathrm{R}\mathcal{I}\mathcal{H}om\left(\mathcal{F},\mathrm{R}\dot{\pi}_{X!!}\left(\mu_X\mathcal{G}|_{\dot{T}^*X}\right)\right).$$

Now, Proposition 2.1.13 gives the following morphism of distinguished triangles where $\mathcal{F}^* = \mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_X)$:

$$\begin{tikzcd} \mathcal{F}^* \otimes \mathrm{R}\dot{\pi}_{X!!}\left(\mu_X\mathcal{G}|_{\dot{T}^*X}\right) \arrow[r] \arrow[d, "\sim"] & \mathcal{F}^* \otimes (\widetilde{\mathbb{K}}_{\Delta_X} \circ \mathcal{G}) \arrow[r] \arrow[d] & \mathcal{F}^* \otimes \mathcal{G} \arrow[r, "+1"] \arrow[d] & {} \\ \mathrm{R}\mathcal{I}\mathcal{H}om\left(\mathcal{F},\mathrm{R}\dot{\pi}_{X!!}(\mu_X\mathcal{G}|_{\dot{T}^*X})\right) \arrow[r] & \mathrm{R}\mathcal{I}\mathcal{H}om(\mathcal{F},\widetilde{\mathbb{K}}_{\Delta_X} \circ \mathcal{G}) \arrow[r] & \mathrm{R}\mathcal{I}\mathcal{H}om(\mathcal{F},\mathcal{G}) \arrow[r, "+1"] & {} \end{tikzcd}$$

The middle vertical arrow is an isomorphism by the following lemma, and hence the right arrow is an isomorphism. $\square$

Lemma 2.3.9. *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathbb{K}_X)$ and $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$. Assume that $\mathcal{F}$ is cohomologically constructible. Then we have an isomorphism*

$$\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_X) \otimes \left(\widetilde{\mathbb{K}}_{\Delta_X} \circ \mathcal{G}\right) \xrightarrow{\sim} \mathrm{R}\mathcal{I}\mathcal{H}om(\mathcal{F},\widetilde{\mathbb{K}}_{\Delta_X} \circ \mathcal{G}).$$

Proof. Let $p_k\colon X \times X \to X$ be the k-th projection ($k = 1,2$). Then we have

$$p_1^{-1}\,\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_X) \otimes p_2^{-1}\mathcal{G} \xrightarrow{\sim} \mathrm{R}\mathcal{H}om(p_1^{-1}\mathcal{F},p_2^{-1}\mathcal{G}) \quad \text{for any } \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X)).$$

Hence we have

$$\begin{aligned}\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_X)\otimes\big(\widetilde{\mathbb{K}}_{\Delta_X}\circ\mathcal{G}\big) &\simeq \mathrm{R}p_{1!!}\big(p_1^{-1}\,\mathrm{R}\mathcal{H}om(\mathcal{F},\mathbb{K}_X)\otimes p_2^{-1}\mathcal{G}\otimes\widetilde{\mathbb{K}}_{\Delta_X}\big)\\ &\simeq \mathrm{R}p_{1!!}\big(\mathrm{R}\mathcal{I}\mathcal{H}om(p_1^{-1}\mathcal{F},p_2^{-1}\mathcal{G})\otimes\widetilde{\mathbb{K}}_{\Delta_X}\big)\\ &\simeq \mathrm{R}p_{1!!}\,\mathrm{R}\mathcal{I}\mathcal{H}om(p_1^{-1}\mathcal{F},p_2^{-1}\mathcal{G}\otimes\widetilde{\mathbb{K}}_{\Delta_X})\\ &\simeq \mathrm{R}\mathcal{I}\mathcal{H}om\Big(\mathcal{F},\mathrm{R}p_{1!!}\big(p_2^{-1}\mathcal{G}\otimes\widetilde{\mathbb{K}}_{\Delta_X}\big)\Big)\simeq \mathrm{R}\mathcal{I}\mathcal{H}om(\mathcal{F},\widetilde{\mathbb{K}}_{\Delta_X}\circ\mathcal{G}).\end{aligned}$$

□

Corollary 2.3.10. *Assume that $i\colon Z\hookrightarrow X$ is a closed immersion and $\mathcal{F}\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$ satisfies the condition*

$$\mathrm{SS}_0(\mathcal{F})\cap T^*_ZX\subset T^*_XX.$$

Then we have an isomorphism

$$i^{-1}\mathcal{F}\otimes\omega_{Z/X}\xrightarrow{\ \sim\ }i^!\mathcal{F}.$$

Proof. We have $i^{-1}\mathcal{F}\otimes\omega_{Z/X}\simeq i^{-1}\mathcal{F}\otimes i^{-1}\,\mathrm{R}\mathcal{H}om(\mathbb{K}_Z,\mathbb{K}_X)\simeq i^{-1}\,\mathrm{R}\mathcal{I}\mathcal{H}om(\mathbb{K}_Z,\mathcal{F})\simeq i^!\mathcal{F}$. □

Lemma 2.3.11. *Let $\Omega\subset\dot{T}^*X$ be an open subset and $\mathcal{K}\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{Y\times\Omega}))$. Assume that*

$$\mathrm{SS}(\mathcal{K})^a\cap\big(T^*Y\times\omega_X(\Omega)\big)=\varnothing,$$

where a denotes the antipodal map. Then

$$(\mathcal{K}\circ\mathrm{K}_{T^*X})|_{Y\times\Omega}=0.$$

Proof. We can easily reduce to the case where $\mathcal{K}\in\mathrm{D^b}(\mathbb{K}_{Y\times\Omega})$. In this case, let us prove that

$$(\mathcal{K}\circ\mathrm{K}_{T^*X})\otimes\widetilde{\mathbb{K}}_p\simeq 0\quad\text{for } p\in Y\times\Omega.$$

We may assume that X, Y are affine and $p=(y_0,x_0;\xi_0)$. We have

$$\mathrm{K}_{T^*X}\otimes\widetilde{\mathbb{K}}_{(x_0,\xi_0)}\simeq \text{“}\varinjlim_{\delta>0,\varepsilon>0}\text{”}\ \mathbb{K}_{F_{\delta,\varepsilon}}[2\dim X],$$

where we have set $F_{\delta,\varepsilon}=\{\delta\geqslant\langle\xi_0,x'-x\rangle>\varepsilon(|x'-x|+|\xi'-\xi|)\}$.
Hence it is enough to show that there exists a neighborhood U of p such that

$$(\mathcal{K}\circ\mathbb{K}_{F_{\delta,\varepsilon}})|_U\simeq 0$$

for $0<\delta\ll\varepsilon\ll 1$. Let p_{ij} be the (i,j)-th projection from $Y\times\Omega\times\Omega$ to $Y\times\Omega$ or $\Omega\times\Omega$. Then we have

$$\mathcal{K}\circ\mathbb{K}_{F_{\delta,\varepsilon}}\simeq\mathrm{R}p_{13!}(p_{12}^{-1}\mathcal{K}\otimes p_{23}^{-1}\,\mathbb{K}_{F_{\delta,\varepsilon}}).$$

For $\mathrm{SS}(F_{\delta,\varepsilon})$ contained in a sufficiently small neighborhood of $(\omega_X(p),-\omega_X(p))$, $\mathrm{SS}\big(p_{12}^{-1}\mathcal{K}\otimes p_{23}^{-1}\,\mathbb{K}_{F_{\delta,\varepsilon}}\big)$ does not intersect $T^*Y\times\{-\langle\xi_0,dx\rangle\}\times T^*\Omega$. Since the map $Y\times\mathrm{Supp}(\mathbb{K}_{F_{\delta,\varepsilon}})\to Y\times\mathbb{R}\times T^*X$ induced by $\langle\xi_0,x\rangle$ is proper, Proposition 5.4.17 in [KS2] implies that $\big(\mathcal{K}\circ\mathbb{K}_{F_{\delta,\varepsilon}}\big)|_U\simeq 0$. □

Proposition 2.3.12. *Let $\mathcal{K}\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_{Y\times X}))$ be a kernel and $\mathcal{F}\in\mathrm{D^b}(\mathrm{I}(\mathbb{K}_X))$. Assume that*

$$\mathrm{SS}(\mathcal{K})^a\cap\big(T^*Y\times\mathrm{SS}_0(\mathcal{F})\big)\subset T^*Y\times T^*_XX.$$

Then we have an isomorphism

$$\mathcal{K}\circ\widetilde{\mathbb{K}}_{\Delta_X}\circ\mathcal{F}\xrightarrow{\ \sim\ }\mathcal{K}\circ\mathcal{F}.$$

Proof. It is enough to show that $\mathcal{K}\circ\bigl(\mathrm{Ker}(\widetilde{\mathbb{K}}_{\Delta_X}\to\mathbb{K}_{\Delta_X})\bigr)\circ\mathcal{F}\simeq 0$.

Let $p\colon Y\times T^*X\to Y\times X$ be the projection. We have

$$\mathrm{SS}\left(p^{-1}\mathcal{K}\right)\subset\Bigl\{((y;\eta),(x,\xi;\xi',0));\ ((y;\eta),(x;\xi'))\in\mathrm{SS}(\mathcal{K})\Bigr\}.$$

Hence,

$$\begin{aligned}&\mathrm{SS}\left(\pi_X^{-1}\mathcal{K}\right)\cap\Bigl(T^*Y\times\omega_X(\mathrm{SS}_0(\mathcal{F})\setminus T^*_XX)\Bigr)\\&\qquad\subset\Bigl\{((y;\eta),(x,\xi;\xi,0));((y;\eta),(x;\xi))\in\mathrm{SS}(\mathcal{K}),(x,\xi)\in\mathrm{SS}_0(\mathcal{F})\setminus T^*_XX\Bigr\}\end{aligned}$$

is empty by assumption. Therefore Lemma 2.3.11 assures that

$$\mathrm{Supp}(p^{-1}\mathcal{K}\circ\mathrm{K}_{T^*X})\cap\bigl(Y\times\mathrm{SS}_0(\mathcal{F})\bigr)\subset Y\times T^*_XX.$$

Let $p_1\colon Y\times T^*X\to Y$ and $p_2\colon Y\times T^*X\to T^*X$ be the projections. Then

$$\begin{aligned}p^{-1}\mathcal{K}\circ\left(\mu_X\mathcal{F}\otimes\widetilde{\mathbb{K}}_{\dot{T}^*X}\right)&\simeq p^{-1}\mathcal{K}\circ\mathrm{K}_{T^*X}\circ\bigl(\mu_X\mathcal{F}\otimes\widetilde{\mathbb{K}}_{\dot{T}^*X}\bigr)\\&\simeq \mathrm{R}p_{1!}\Bigl((p^{-1}\mathcal{K}\circ\mathrm{K}_{T^*X})\otimes p_2^{-1}(\mu_X\mathcal{F}\otimes\widetilde{\mathbb{K}}_{\dot{T}^*X})\Bigr)\simeq 0.\end{aligned}$$

This proves the proposition since $p^{-1}\mathcal{K}\circ(\mu_X\mathcal{F}\otimes\widetilde{\mathbb{K}}_{\dot{T}^*X})\simeq\mathcal{K}\circ\mathrm{R}\pi_{X!!}(\mu_X\mathcal{F}\otimes\widetilde{\mathbb{K}}_{\dot{T}^*X})$ by Lemma 1.1.6 (iii), and $\mathrm{R}\pi_{X!!}\bigl(\mu_X\mathcal{F}\otimes\widetilde{\mathbb{K}}_{\dot{T}^*X}\bigr)\simeq\mathrm{Ker}\bigl(\widetilde{\mathbb{K}}_{\Delta_X}\to\mathbb{K}_{\Delta_X}\bigr)\circ\mathcal{F}$ by Proposition 2.1.13 (iii). □

2.4. **Functorial properties of microlocalization.** To study the functorial behavior of the functor μ_X, it is convenient to introduce various transfer kernels. They will be used exclusively inside the proofs in order to keep notations as simple as possible. In the sequel, we frequently use Lemma 1.1.6 without mentioning it.

Let $f\colon X\to Y$ be a morphism of manifolds. Let us recall the commutative diagram:

$$\begin{tikzcd} T^*X \arrow[d, "\pi_X"'] & T^*Y\underset{Y}{\times}X \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d] & T^*Y \\ X \arrow[r, "f"] & Y. & \end{tikzcd}$$

We have $f_d^*\omega_X=f_\pi^*\omega_Y$. Consider the maps

$$\begin{aligned}(T^*Y\underset{Y}{\times}X)\times X&\xrightarrow{f_d\times\mathrm{id}_X}T^*X\times X,\\(T^*Y\underset{Y}{\times}X)\times Y&\xrightarrow{f_\pi\times\mathrm{id}_Y}T^*Y\times Y,\\T^*Y\times X&\xrightarrow{\mathrm{id}_{T^*Y}\times f}T^*Y\times Y.\end{aligned}$$

They define morphisms

$$\begin{aligned}\Gamma(T^*X\underset{X}{\times}X,\Omega^1_{T^*X\times X})&\to\Gamma(T^*Y\underset{Y}{\times}X,\Omega^1_{(T^*Y\underset{Y}{\times}X)\times X}),\\\Gamma(T^*Y,\Omega^1_{T^*Y\times Y})&\to\Gamma(T^*Y\underset{Y}{\times}X,\Omega^1_{(T^*Y\underset{Y}{\times}X)\times Y}),\\\Gamma(T^*Y,\Omega^1_{T^*Y\times Y})&\to\Gamma(T^*Y\underset{Y}{\times}X,\Omega^1_{T^*Y\times X}).\end{aligned}$$

We denote by $\sigma_{Y\leftarrow X}, \sigma_{X\to Y}$ and $\sigma_{X|Y}$ the images of the section σ_X, σ_Y and σ_Y (defined in 2.1.5), respectively. We set

$$\begin{aligned}
\mathrm{L}_{Y\leftarrow X} &= \mathcal{L}_{\sigma_{Y\leftarrow X}}((T^*Y \underset{Y}{\times} X) \underset{X}{\times} X, (T^*Y \underset{Y}{\times} X) \times X),\\
\mathrm{L}_{X\to Y} &= \mathcal{L}_{\sigma_{X\to Y}}((T^*Y \underset{Y}{\times} X) \underset{Y}{\times} Y, (T^*Y \underset{Y}{\times} X) \times Y),\\
\mathrm{L}_{X|Y} &= \mathcal{L}_{\sigma_{X|Y}}(T^*Y \underset{Y}{\times} X, T^*Y \times X).
\end{aligned}$$

Note that if $f = \mathrm{id}_X \colon X \to X$, then these three kernels coincide and are isomorphic to L_X.

Lemma 2.4.1. *Let $f\colon X \to Y$ be a morphism of manifolds. There are natural isomorphisms*

(i) $\mathrm{L}_X \simeq \mathrm{R}\,(\mathrm{id}_{T^*X} \times \pi_X)_{!!}\,\mathrm{K}_{T^*X}$,
(ii) $(f_d \times \mathrm{id}_X)^{-1}\,\mathrm{L}_X \simeq \mathrm{L}_{Y\leftarrow X}$,
(iii) $\mathrm{L}_{X|Y} \simeq\ (\mathrm{id}_{T^*Y} \times f)^{-1}\,\mathrm{L}_Y$,
(iv) $\mathrm{K}_{T^*Y} \underset{T^*Y}{\circ} \mathrm{L}_{X|Y} \simeq \mathrm{L}_{X|Y}$,
(v) $\mathrm{R}(f_\pi \times \mathrm{id}_X)_{!!}\,\mathrm{L}_{Y\leftarrow X} \longrightarrow \mathrm{K}_{T^*Y} \underset{T^*Y}{\circ} \mathrm{R}(f_\pi \times \mathrm{id}_X)_{!!}\,\mathrm{L}_{Y\leftarrow X} \xrightarrow{\sim} \mathrm{L}_{X|Y}$,
(vi) $\mathrm{R}(f_\pi \times \mathrm{id}_X)_{!!}\,\mathrm{L}_{Y\leftarrow X} \xrightarrow{\sim} \mathrm{L}_{X|Y}$ *if f is smooth,*
(vii) $(f_\pi \times \mathrm{id}_Y)^{-1}\,\mathrm{L}_Y \simeq \mathrm{L}_{X\to Y}$.
(viii) *Moreover, there is a morphism* $\mathrm{R}(\mathrm{id}_{T^*Y\underset{Y}{\times}X} \times f)_{!!}\,\mathrm{L}_{Y\leftarrow X} \longrightarrow \mathrm{L}_{X\to Y}$ *which is an isomorphism if f is smooth.*

The results easily follow from the first part of the paper.

Theorem 2.4.2 (proper direct image)**.** *Let $f\colon X \to Y$ be a morphism of manifolds and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_X))$. Then*

(i) *we have a natural morphism and a natural isomorphism*

$$\mathrm{R}f_{\pi!!}{f_d}^{-1}\mu_X\mathcal{F} \longrightarrow \mathrm{K}_{T^*Y} \circ \mathrm{R}f_{\pi!!}{f_d}^{-1}\mu_X\mathcal{F} \xrightarrow{\sim} \mu_Y\,(\mathrm{R}f_{!!}\mathcal{F}),$$

(ii) *if f is smooth we get an isomorphism*

$$\mathrm{R}f_{\pi!!}{f_d}^{-1}\mu_X\mathcal{F} \xrightarrow{\sim} \mu_Y\,(\mathrm{R}f_{!!}\mathcal{F}).$$

Proof. We have ${f_d}^{-1}\mu_X\mathcal{F} \simeq \mathrm{L}_{Y\leftarrow X} \circ\mathcal{F}$ by Lemma 2.4.1 (ii), and a natural morphism by Lemma 2.4.1 (v),

$$\mathrm{R}(f_\pi \times \mathrm{id}_X)_{!!}\,\mathrm{L}_{Y\leftarrow X} \longrightarrow \mathrm{K}_{T^*Y} \underset{T^*Y}{\circ} \mathrm{R}(f_\pi \times \mathrm{id}_X)_{!!}\,\mathrm{L}_{Y\leftarrow X} \xrightarrow{\sim} \mathrm{L}_{X|Y}.$$

However $(\mathrm{R}(f_\pi \times \mathrm{id}_X)_{!!}L_{Y\leftarrow X}) \circ \mathcal{F} \simeq \mathrm{R}f_{\pi!!}{f_d}^{-1}\mu_X\mathcal{F}$ and $L_{X|Y} \circ \mathcal{F} \simeq \mu_Y\,(\mathrm{R}f_{!!}\mathcal{F})$. Hence we get natural morphisms

$$\mathrm{R}f_{\pi!!}{f_d}^{-1}\mu_X\mathcal{F} \longrightarrow \mathrm{K}_{T^*Y} \circ \mathrm{R}f_{\pi!!}{f_d}^{-1}\mu_X\mathcal{F} \xrightarrow{\sim} \mu_Y\,(\mathrm{R}f_{!!}\mathcal{F}),$$

which are isomorphisms if f is smooth by Lemma 2.4.1 (vi). □

Proposition 2.4.3 (inverse image)**.** *Let $f\colon X \to Y$ be a morphism of manifolds and $\mathcal{G} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{K}_Y))$. Then*

(i) *we have a natural morphism*

$${f_d}^{-1}\mu_X(f^{-1}\mathcal{G}) \longrightarrow {f_\pi}^{-1}\mu_Y\mathcal{G},$$

which is an isomorphism if f is smooth,

(ii) *we have a natural morphism*

$$\mu_X(f^{-1}\mathcal{G}) \longrightarrow \mathrm{R}f_{d*}{f_\pi}^{-1}\mu_Y\mathcal{G}.$$

Proof. We have

$${f_d}^{-1}\mu_X\left(f^{-1}\mathcal{G}\right) \simeq \mathrm{L}_{Y\leftarrow X}\circ f^{-1}\mathcal{G} \qquad \text{and} \qquad {f_\pi}^{-1}\mu_Y\mathcal{G} \simeq \mathrm{L}_{X\to Y}\circ\mathcal{G}.$$

Since $\mathrm{L}_{Y\leftarrow X}\circ f^{-1}\mathcal{G} \simeq \left(\mathrm{R}\,(\mathrm{id}_{T^*Y\underset{Y}{\times}X}\times f)_{!!}\,\mathrm{L}_{Y\leftarrow X}\right)\circ\mathcal{G}$, we deduce a morphism by Lemma 2.4.1 (viii):

$${f_d}^{-1}\mu_X\left(f^{-1}\mathcal{G}\right) \simeq \left(\mathrm{R}\,(\mathrm{id}_{T^*Y\underset{Y}{\times}X}\times f)_{!!}\,\mathrm{L}_{Y\leftarrow X}\right)\circ\mathcal{G} \longrightarrow \mathrm{L}_{X\leftarrow Y}\circ\mathcal{G} \simeq {f_\pi}^{-1}\mu_Y\mathcal{G},$$

which is an isomorphism whenever f is smooth. By adjunction we get then the inverse image morphism $\mu_X(f^{-1}\mathcal{G}) \longrightarrow \mathrm{R}f_{d*}{f_\pi}^{-1}\mu_Y\mathcal{G}$. □

Theorem 2.4.4 (embedding case)**.** *Let $f\colon X \hookrightarrow Y$ be a closed embedding. Then the following statements hold: for $\mathcal{G} \in \mathrm{D^b}\left(\mathrm{I}\left(\mathbb{K}_Y\right)\right)$.*

(i) *we have a natural morphism*

$$\mathrm{R}f_{d!!}{f_\pi}^{-1}\mu_Y(\mathcal{G}) \longrightarrow \mu_X(f^{-1}\mathcal{G}),$$

(ii) *if X is non characteristic for $\mathcal{G}$* (i.e. $\mathrm{SS}_0(\mathcal{G})\cap T^*_XY \subset T^*_YY$)*, then the morphism in* (i) *is an isomorphism and* $\mathrm{SS}_0(f^{-1}\mathcal{G}) \subset f_d{f_\pi}^{-1}\,\mathrm{SS}_0(\mathcal{G})$*.*

Proof. (i) Consider the following diagrams

$$\begin{tikzcd} T^*X & T^*Y\underset{Y}{\times}X \arrow[l, "f_d"'] \arrow[d, hook, "f_\pi"] \\ & T^*Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} X & T^*Y\underset{Y}{\times}X \arrow[l, "p"'] \arrow[d, equal] & \left(T^*Y\underset{Y}{\times}X\right)\times X \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, hook, "f'"] & X \arrow[d, hook, "f"] \\ & T^*Y\underset{Y}{\times}X & \left(T^*Y\underset{Y}{\times}X\right)\times Y \arrow[l, "p_1'"'] \arrow[r, "p_2'"] & Y. \end{tikzcd}$$

We have

$${f_d}^{-1}\mu_X\left(f^{-1}\mathcal{G}\right) \simeq \mathrm{L}_{Y\leftarrow X}\circ f^{-1}\mathcal{G} \qquad \text{and} \qquad {f_\pi}^{-1}\mu_Y\mathcal{G} \simeq \mathrm{L}_{X\to Y}\circ\mathcal{G}.$$

Since f is a closed immersion, f_d is smooth and we get

$${f_d}^{!}\mu_X\left(f^{-1}\mathcal{G}\right) \simeq \left(\mathrm{L}_{Y\leftarrow X}\circ f^{-1}\mathcal{G}\right)\otimes\omega_{T^*Y\underset{Y}{\times}X/T^*X}.$$

The cotangent bundles being canonically orientable, we have

$$\omega_{T^*Y\underset{Y}{\times}X/T^*X} \simeq p^{-1}\omega_{X/Y}[2(\dim Y - \dim X)] \simeq p^{-1}\omega_{X/Y}^{\otimes -1},$$

where $p\colon T^*Y\underset{Y}{\times}X \to X$ is the projection. Hence we get

$${f_d}^{!}\mu_X\left(f^{-1}\mathcal{G}\right) \simeq \left(\mathrm{L}_{Y\leftarrow X}\circ f^{-1}\mathcal{G}\right)\otimes p^{-1}\omega_{X/Y}^{\otimes -1}.$$

Now since f' is a closed immersion, $\mathrm{L}_{Y\leftarrow X} \simeq f'^{\,!}\,\mathrm{L}_{X\to Y}$ using Proposition 1.3.7, which induces a morphism

$$f'^{-1}\,\mathrm{L}_{X\to Y} \to \mathrm{L}_{Y\leftarrow X}\otimes\omega^{\otimes -1}_{X\times(T^*Y\underset{Y}{\times}X)/Y\times(T^*Y\underset{Y}{\times}X)} \simeq \mathrm{L}_{Y\leftarrow X}\otimes p_2^{-1}\omega^{\otimes -1}_{X/Y} \simeq \mathrm{L}_{Y\leftarrow X}\otimes p_1^{-1}p^{-1}\omega^{\otimes -1}_{X/Y}.$$

Then the preceding morphism together with the adjunction morphism $\mathrm{id} \to \mathrm{R}f'_*f'^{-1} \simeq \mathrm{R}f'_{!!}f'^{-1}$ provides a morphism

$$\begin{aligned} {f_\pi}^{-1}\mu_Y\mathcal{G} &\simeq \mathrm{L}_{X\to Y}\circ\mathcal{G} = \mathrm{R}p'_{1!!}(\mathrm{L}_{X\to Y}\otimes {p'_2}^{-1}\mathcal{G}) \simeq \mathrm{R}p'_{1!!}\,\mathrm{R}f'_{!!}f'^{-1}(\mathrm{L}_{X\to Y}\otimes {p'_2}^{-1}\mathcal{G}) \\ &\to \mathrm{R}p_{1!!}(\mathrm{L}_{Y\leftarrow X}\otimes p_1^{-1}p^{-1}\omega^{\otimes -1}_{X/Y}\otimes p_2^{-1}f^{-1}\mathcal{G}) \simeq (\mathrm{L}_{Y\leftarrow X}\circ f^{-1}\mathcal{G})\otimes p^{-1}\omega^{\otimes -1}_{X/Y}. \end{aligned}$$

Finally we obtain a morphism

$$f_\pi{}^{-1}\mu_Y\mathcal{G} \longrightarrow (\mathrm{L}_{Y\leftarrow X}\circ f^{-1}\mathcal{G})\otimes p^{-1}\omega_{X/Y}^{\otimes -1} \simeq f_d{}^{-1}\mu_X\left(f^{-1}\mathcal{G}\right)\otimes p^{-1}\omega_{X/Y}^{\otimes -1} \simeq f_d{}^{!}\mu_X\left(f^{-1}\mathcal{G}\right),$$

and by adjunction the desired morphism

$$\mathrm{R}f_{d!!}f_\pi{}^{-1}\mu_Y(\mathcal{G}) \longrightarrow \mu_X(f^{-1}\mathcal{G}).$$

(ii) Assume now that X is non characteristic for $\mathcal{G}$. By induction we may assume that X is a hypersurface in Y. For $p\in T^*X$, let us show that $\mathrm{R}f_{d!!}f_\pi{}^{-1}\mu_Y(\mathcal{G})\otimes\widetilde{\mathbb{K}}_p \xrightarrow{\sim} \mu_X(f^{-1}\mathcal{G})\otimes\widetilde{\mathbb{K}}_p$.

Assume first that $p\in T^*_XX$. Since X is non characteristic for $\mathcal{G}$ we get

$$\begin{aligned}\mathrm{R}f_{d!!}f_\pi^{-1}\mu_Y\mathcal{G}\otimes\widetilde{\mathbb{K}}_p &\simeq \mathrm{R}f_{d!!}\big(f_\pi^{-1}\mu_Y\mathcal{G}\otimes\widetilde{\mathbb{K}}_{T^*_XY}\big)\otimes\widetilde{\mathbb{K}}_p \simeq \mathrm{R}f_{d!!}\big(f_\pi^{-1}\big(\mu_Y\mathcal{G}\otimes\widetilde{\mathbb{K}}_{T^*_YY}\big)\big)\otimes\widetilde{\mathbb{K}}_p\\ &\simeq \mathrm{R}f_{d!!}\big(f_\pi^{-1}(\pi_Y^{-1}\mathcal{G}\otimes\widetilde{\mathbb{K}}_{T^*_YY})\big)\otimes\widetilde{\mathbb{K}}_p \simeq \mathrm{R}f_{d!!}\big(f_d^{-1}\pi_X^{-1}f^{-1}\mathcal{G}\otimes\widetilde{\mathbb{K}}_{T^*_YY\underset{Y}{\times}X}\big)\otimes\widetilde{\mathbb{K}}_p\\ &\simeq \pi_X^{-1}f^{-1}\mathcal{G}\otimes\mathrm{R}f_{d!!}\,\widetilde{\mathbb{K}}_{T^*_YY\underset{Y}{\times}X}\otimes\widetilde{\mathbb{K}}_p \simeq \pi_X^{-1}f^{-1}\mathcal{G}\otimes\widetilde{\mathbb{K}}_p\\ &\simeq \mu_Xf^{-1}\mathcal{G}\otimes\widetilde{\mathbb{K}}_p\,.\end{aligned}$$

Assume now that $p\notin T^*_XX$. Consider the following diagram

$$\begin{tikzcd}
T^*X\times X \arrow[r, hook, "f_1"] & T^*X\times Y \arrow[d, "q_1"'] \arrow[rr, bend left, "q_2"] \arrow[dr, phantom, "\Box"] & (T^*Y\underset{Y}{\times}X)\times Y \arrow[l, "r"'] \arrow[r, "p'_2"'] \arrow[d, "p'_1"] & Y\\
& T^*X & T^*Y\underset{Y}{\times}X \arrow[l, "f_d"] &
\end{tikzcd}$$

Note that

$$\mathrm{R}f_{d!!}f_\pi^{-1}\mu_Y\mathcal{G}\simeq(\mathrm{R}r_{!!}\,\mathrm{L}_{X\to Y})\circ\mathcal{G} \qquad\text{and}\qquad \mu_Xf^{-1}\mathcal{G}\simeq\mathrm{L}_X\circ f^{-1}\mathcal{G}\simeq(\mathrm{R}f_{1!!}\,\mathrm{L}_X)\circ\mathcal{G}.$$

Hence we have to prove that

$$\big(\mathrm{R}r_{!!}\,\mathrm{L}_{X\to Y}\otimes\widetilde{\mathbb{K}}_p\big)\circ\mathcal{G}\simeq\big(\mathrm{R}f_{1!!}\,\mathrm{L}_X\otimes\widetilde{\mathbb{K}}_p\big)\circ\mathcal{G}.$$

Here we identify $p\in T^*X$ with $\big(p,f(\pi_X(p))\big)\in T^*X\times Y$. Take a local coordinate system $(t,x)=(t,x_1,\dots,x_n)$ of Y such that X is given by $t=0$ and denote by (t,x,τ,ξ) and (x,ξ) the associated coordinates on T^*Y and T^*X, respectively. Set $p=(0,\xi_0)$. Let $((x,\tau,\xi),(t',x'))$ be the coordinates of $(T^*Y\underset{Y}{\times}X)\times Y$. Then $r((x,\tau,\xi),(t',x'))=((x,\xi),(t',x'))$. We have

$$\mathrm{R}r_{!!}\,\mathrm{L}_{X\to Y}\otimes\widetilde{\mathbb{K}}_p\simeq\mathrm{R}r_{!!}\Big(\underset{\varepsilon>0,\,R>0}{\text{“}\varinjlim\text{”}}\ \mathbb{K}_{\{\tau t'+\langle\xi_0,x'-x\rangle>\varepsilon(|t'|+|x'-x|),\,|\tau|<R\}}[\dim Y]\Big)\otimes\widetilde{\mathbb{K}}_p\,.$$

Since the fiber of $\{\tau t'+\langle\xi_0,x'-x\rangle>\varepsilon(|t'|+|x'-x|),\ |\tau|<R\}$ over $((x,\xi),t',x')$ is a non-empty open interval if $R|t'|+\langle\xi_0,x'-x\rangle>\varepsilon(|t'|+|x'-x|)$, and empty otherwise, we obtain

$$\mathrm{R}r_{!!}\,\mathrm{L}_{X\to Y}\otimes\widetilde{\mathbb{K}}_p\simeq\Big(\underset{\varepsilon>0,\,R>0}{\text{“}\varinjlim\text{”}}\ \mathbb{K}_{\{R|t'|+\langle\xi_0,x'-x\rangle>\varepsilon(|t'|+|x'-x|)\}}[\dim Y-1]\Big)\otimes\widetilde{\mathbb{K}}_p\,.$$

Therefore

$$\big(\mathrm{R}r_{!!}\,\mathrm{L}_{X\to Y}\otimes\widetilde{\mathbb{K}}_p\big)\circ\mathcal{G}\simeq\Big(\underset{\varepsilon>0,\,R>0}{\text{“}\varinjlim\text{”}}\ \mathbb{K}_{\{R|t'|+\langle\xi_0,x'-x\rangle>\varepsilon|x'-x|\}}[\dim X]\otimes\widetilde{\mathbb{K}}_p\Big)\circ\mathcal{G}.$$

On the other hand we have

$$\big(\mathrm{R}f_{1!!}\,\mathrm{L}_X \otimes \widetilde{\mathbb{K}}_p\big) \circ \mathcal{G} \simeq \Big(\mathrm{R}f_{1!!}\big(\text{``}\varinjlim_{\varepsilon>0}\text{''}\, \mathbb{K}_{\{\langle\xi_0,x'-x>\varepsilon|x'-x|\}}[\dim X]\big) \otimes \widetilde{\mathbb{K}}_p\Big) \circ \mathcal{G}$$
$$\simeq \Big(\text{``}\varinjlim_{\varepsilon>0}\text{''}\, \mathbb{K}_{\{\langle\xi_0,x'-x\rangle>\varepsilon|x'-x|,\, t'=0\}}[\dim X] \otimes \widetilde{\mathbb{K}}_p\Big) \circ \mathcal{G}.$$

Hence it is enough to show that

$$\big(\ \text{``}\varinjlim_{\varepsilon>0,\, R>0}\text{''}\ \ \mathbb{K}_{\{R|t'|+\langle\xi_0,x'-x\rangle>\varepsilon|x'-x|,\, 0<t'\}} \otimes \widetilde{\mathbb{K}}_p\,\big) \circ \mathcal{G} \simeq 0.$$

Let us set $U_{\varepsilon,\delta,R} = \{R\,t' + \langle\xi_0, x'-x\rangle > |x-x|,\ 0 < t' \le \delta\}$. For ε, δ sufficiently small and R sufficiently large, $\mathrm{SS}(\mathbb{K}_{U_{\varepsilon,\delta,R}})$ is contained in a sufficiently small neighborhood of $-Rdt' + \langle\xi_0, d(x-x')\rangle$ on a neighborhood of p. Hence we obtain

$$\mathrm{SS}(\mathbb{K}_{U_{\varepsilon,\delta,R}})^a \cap T^*(T^*X) \times \mathrm{SS}_0(\mathcal{G}) \subset T^*(T^*X) \times T^*_Y X \quad \text{on a neighborhood of } p \text{ for } R \gg 0.$$

Therefore Proposition 2.3.12 implies

$$\big(\mathbb{K}_{U_{\varepsilon,\delta,R}} \circ \widetilde{\mathbb{K}}_{\Delta_Y}\big) \circ \mathcal{G} \simeq \mathbb{K}_{U_{\varepsilon,\delta,R}} \circ \mathcal{G} \quad \text{on a neighborhood of } p \text{ for } R \gg 0.$$

Hence we have reduced the problem to

$$\big(\ \text{``}\varinjlim_{\varepsilon>0,\,\delta>0\, R>0}\text{''}\ \ \mathbb{K}_{U_{\varepsilon,\delta,R}} \otimes \widetilde{\mathbb{K}}_p\big) \circ \widetilde{\mathbb{K}}_{\Delta_Y} \simeq 0.$$

Consider the projection on the first and third factors

$$h\colon T^*X \times Y \times Y \longrightarrow T^*X \times Y \qquad i.e.\ ((x;\xi),(t',x'),(t'',x'')) \mapsto ((x;\xi),(t'',x'')).$$

Then

$$\big(\ \text{``}\varinjlim_{\varepsilon>0,\,\delta>0\, R>0}\text{''}\ \ \mathbb{K}_{U_{\varepsilon,\delta,R}} \otimes \widetilde{\mathbb{K}}_p\big) \circ \widetilde{\mathbb{K}}_{\Delta_Y} \simeq \mathrm{R}h_{!!}\big(\ \text{``}\varinjlim_{\varepsilon>0,\,\delta>0\, R>0}\text{''}\ \ \mathbb{K}_{V_{\varepsilon,\delta,R}}\big) \otimes \widetilde{\mathbb{K}}_p,$$

where $V_{\varepsilon,\delta,R} = \{R\,t' + \langle\xi_0, x'-x\rangle > \varepsilon|x'-x|,\ 0 < t' \le \delta,\ |x'-x''| \le \delta,\ |t'-t''| \le \delta\}$. This vanishes by the following lemma. ☐

Sublemma 2.4.5. *Let $(t,t',x,y) = (t,t',x_1,\dots,x_n,y_1,\dots,y_n)$ be the coordinates of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, and let $h\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}\times\mathbb{R}^n$ be the projection, $h(t,t',x,y) = (t',y)$. For $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\delta > 0$, set $V_\delta = \{(t,t',x,y);\ t + \langle\xi_0,x\rangle > |x|,\ |x-y| \le \delta, 0 < t \le \delta,\ |t-t'| \le \delta\}$. Then*

$$\mathrm{Supp}\big(\mathrm{R}h_!\ \mathbb{K}_{V_\delta}\big) \not\ni 0.$$

Proof. Let us decompose h into $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \xrightarrow{h_1} \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \xrightarrow{h_2} \mathbb{R}\times\mathbb{R}^n$, where $h_1(t,t',x,y) = (t',x,y)$ and $h_2(t',x,y) = (t',y)$. When $|x-y| \le \delta$, the fiber $V_\delta \cap h_1^{-1}(t',x,y)$ is $\{t; \max(0,\ |x| - \langle\xi_0,x\rangle) < t \le \min(\delta, t'+\delta),\ t'-\delta \le t\}$. Hence, setting

$$W_\delta = \{(t',x,y); \max(0,\ |x| - \langle\xi_0,x\rangle) < t'-\delta \le \min(\delta, t'+\delta),\ |x-y| \le \delta\},$$

we have $\mathrm{R}h_{1!}\ \mathbb{K}_{V_\delta} \simeq \mathbb{K}_{W_\delta}$. Since $\mathrm{Supp}(\mathbb{K}_{W_\delta}) \subset \{(t',x,y); \delta \le t'\}$, we obtain

$$\mathrm{Supp}(\mathrm{R}h_!\ \mathbb{K}_{V_\delta}) \subset \{(t',y); \delta \le t'\}.$$

☐

2.5. **Microlocal convolution of kernels.** Let X, Y and Z be manifolds, and let p_{ij} be the (i,j)-th projection from $T^*X \times T^*Y \times T^*Z$. As usual, denote by $a\colon T^*X \to T^*X$ the antipodal map. Then define the antipodal projection p_{12}^a by

$$p_{12}^a\colon T^*X \times T^*Y \times T^*Z \xrightarrow{p_{12}} T^*X \times T^*Y \xrightarrow{\mathrm{id}\times a} T^*X \times T^*Y.$$

For $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_{T^*X\times T^*Y}))$ and $\mathcal{G} \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_{T^*Y\times T^*Z}))$, we set

$$\mathcal{F} \overset{a}{\circ} \mathcal{G} = \mathrm{R}p_{13!!}\big(p_{12}^{a-1}\mathcal{F} \otimes p_{23}^{-1}\mathcal{G}\big).$$

In an analogous way, for $S_1 \subset T^*X \times T^*Y$ and $S_2 \subset T^*Y \times T^*Z$, we set

$$S_1 \underset{T^*Y}{\overset{a}{\times}} S_2 = p_{12}^{a-1}(S_1) \cap p_{23}^{-1}(S_2) \subset T^*X \times T^*Y \times T^*Z.$$

Now we are ready to state the main theorem:

Theorem 2.5.1 (Microlocal convolution of kernels)**.** *Let $\mathcal{K}_1 \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_{X\times Y}))$ and $\mathcal{K}_2 \in \mathrm{D^b}(\mathrm{I}(\mathbb{K}_{Y\times Z}))$.*

(i) *There is a natural morphism*

$$\mu_{X\times Y}\mathcal{K}_1 \overset{a}{\circ} \mu_{Y\times Z}\mathcal{K}_2 \longrightarrow \mu_{X\times Z}(\mathcal{K}_1 \circ \mathcal{K}_2). \tag{2.9}$$

(ii) *Assume the non-characteristic condition*

$$\mathrm{SS}_0(\mathcal{K}_1) \underset{T^*Y}{\overset{a}{\times}} \mathrm{SS}_0(\mathcal{K}_2) \cap (T^*_X X \times T^*Y \times T^*_Z Z) \subset T^*_X X \times T^*_Y Y \times T^*_Z Z, \tag{2.10}$$

Then (2.9) *is an isomorphism outside*

$$\overline{p_{13}\big(\mathrm{SS}_0(\mathcal{K}_1) \underset{T^*Y}{\overset{a}{\times}} \mathrm{SS}_0(\mathcal{K}_2) \cap T^*X \times T^*_Y Y \times T^*Z\big)}.$$

Proof. (a) We shall first construct the morphism. Consider the manifolds $\mathcal{X}_1 = X \times Y$, $\mathcal{X}_2 = Y \times Z$ and $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 = X \times Y \times Y \times Z$ together with the diagonal embedding

$$\mathcal{Y} := X \times Y \times Z \overset{j}{\hookrightarrow} \mathcal{X}.$$

Denote by $\mathcal{Z} = X \times Z$, and let $q_{13}\colon \mathcal{Y} \to \mathcal{Z}$ be the projection. The map

$$T^*\mathcal{Y} \hookrightarrow \mathcal{Y} \underset{\mathcal{X}}{\times} T^*\mathcal{X} \qquad \text{given by} \qquad (x,y,z;\xi,\eta,\zeta) \mapsto (x,y,y,z;\xi,-\eta,\eta,\zeta)$$

defines the cartesian square in the following commutative diagram:

$$\begin{tikzcd}
T^*\mathcal{Y} \arrow[r, hook] \arrow[d] \arrow[rr, bend left, "p"] \arrow[dd, bend right, "q"'] \arrow[dr, phantom, "\square"] & \mathcal{Y} \underset{\mathcal{X}}{\times} T^*\mathcal{X} \arrow[r, hook, "j_\pi"] \arrow[d, "j_d"'] & T^*\mathcal{X} \\
\mathcal{Y} \underset{\mathcal{Z}}{\times} T^*\mathcal{Z} \arrow[r, hook, "q_{13d}"'] \arrow[d, "q_{13\pi}"] & T^*\mathcal{Y} & \\
T^*\mathcal{Z} & &
\end{tikzcd}$$

By Proposition 2.1.14, we have an isomorphism

$$\mathrm{K}_{T^*\mathcal{X}} \circ (\mu_{\mathcal{X}_1}\mathcal{K}_1 \boxtimes \mu_{\mathcal{X}_2}\mathcal{K}_2) \simeq \mu_{\mathcal{X}}(\mathcal{K}_1 \boxtimes \mathcal{K}_2).$$

By Theorem 2.4.4 we have a morphism

$$\mathrm{R}j_{d!!}j_\pi^{-1}\mu_{\mathcal{X}}(\mathcal{K}_1 \boxtimes \mathcal{K}_2) \longrightarrow \mu_{\mathcal{Y}}(j^{-1}(\mathcal{K}_1 \boxtimes \mathcal{K}_2)). \tag{2.11}$$

Since q_{13} is smooth we also have an isomorphism by Theorem 2.4.2 (ii)

$$\mathrm{R}q_{13\pi!!}q_{13d}^{-1}\mu_{\mathcal{Y}}(j^{-1}(\mathcal{K}_1\boxtimes\mathcal{K}_2)) \xrightarrow{\sim} \mu_{\mathcal{Z}}(\mathrm{R}q_{13!!}j^{-1}(\mathcal{K}_1\boxtimes\mathcal{K}_2)) \simeq \mu_{\mathcal{Z}}(\mathcal{K}_1\circ\mathcal{K}_2).$$

Hence we get a morphism

$$\mathrm{R}q_{!!}p^{-1}\Big(\mathrm{K}_{T^*\mathcal{X}}\circ(\mu_{\mathcal{X}_1}\mathcal{K}_1\boxtimes\mu_{\mathcal{X}_2}\mathcal{K}_2)\Big) \to \mu_{\mathcal{Z}}(\mathcal{K}_1\circ\mathcal{K}_2). \tag{2.12}$$

Hence we obtain

$$\begin{aligned}\mu_{\mathcal{X}_1}\mathcal{K}_1 \overset{a}{\circ} \mu_{\mathcal{X}_2}\mathcal{K}_2 &\simeq \mathrm{R}q_{!!}p^{-1}(\mu_{\mathcal{X}_1}\mathcal{K}_1\boxtimes\mu_{\mathcal{X}_2}\mathcal{K}_2)\\ &\to \mathrm{R}q_{!!}p^{-1}\Big(\mathrm{K}_{T^*\mathcal{X}}\circ(\mu_{\mathcal{X}_1}\mathcal{K}_1\boxtimes\mu_{\mathcal{X}_2}\mathcal{K}_2)\Big) \to \mu_{\mathcal{Z}}(\mathcal{K}_1\circ\mathcal{K}_2).\end{aligned}$$

(b) By Theorem 2.4.4, (2.11) is an isomorphism under the non-characteristic hypothesis, and hence (2.12) is also an isomorphism under the same hypothesis.

Therefore in order to show (ii), it is enough to show that

$$\begin{aligned}&\mu_{\mathcal{X}_1}\mathcal{K}_1 \overset{a}{\circ} \mu_{\mathcal{X}_2}\mathcal{K}_2 \simeq \mathrm{R}q_{!!}p^{-1}\Big(\mathrm{K}_{T^*\mathcal{X}}\circ(\mu_{\mathcal{X}_1}\mathcal{K}_1\boxtimes\mu_{\mathcal{X}_2}\mathcal{K}_2)\Big)\\ &\qquad\text{outside } p_{13}\big(\mathrm{SS}_0(\mathcal{K}_1) \underset{T^*Y}{\overset{a}{\times}} \mathrm{SS}_0(\mathcal{K}_2)\cap T^*X\times T^*_YY\times T^*Z\big).\end{aligned} \tag{2.13}$$

First note that

$$\begin{aligned}\mu_{\mathcal{X}_1}\mathcal{K}_1 \overset{a}{\circ} \mu_{\mathcal{X}_2}\mathcal{K}_2 &\simeq (\mathrm{K}_{T^*\mathcal{X}_1}\circ\mu_{\mathcal{X}_1}\mathcal{K}_1) \overset{a}{\circ} (\mathrm{K}_{T^*\mathcal{X}_1}\circ\mu_{\mathcal{X}_2}\mathcal{K}_2)\\ &\simeq \big(\mathrm{K}_{T^*\mathcal{X}_1}\underset{T^*Y}{\circ}\mathrm{K}_{T^*\mathcal{X}_2}\big)\circ\big(\mu_{\mathcal{X}_1}\mathcal{K}_1\boxtimes\mu_{\mathcal{X}_2}\mathcal{K}_2\big).\end{aligned}$$

Consider the diagram

$$\begin{tikzcd} & T^*\mathcal{Y}\times T^*\mathcal{X} \arrow[dl, "q'=(q,\mathrm{id})"'] \arrow[dr, "p'=(p,\mathrm{id})"] & \\ T^*\mathcal{Z}\times T^*\mathcal{X} & & T^*\mathcal{X}\times T^*\mathcal{X} \end{tikzcd}$$

Then we have

$$\mathrm{R}q_{!!}p^{-1}\Big(\mathrm{K}_{T^*\mathcal{X}}\circ(\mu_{\mathcal{X}_1}\mathcal{K}_1\boxtimes\mu_{\mathcal{X}_2}\mathcal{K}_2)\Big) \simeq \big(\mathrm{R}q'_{!!}p'^{-1}\,\mathrm{K}_{T^*\mathcal{X}}\big)\circ(\mu_{\mathcal{X}_1}\mathcal{K}_1\boxtimes\mu_{\mathcal{X}_2}\mathcal{K}_2).$$

Using Proposition 1.3.3 and Corollary 1.3.5, we have

$$\mathrm{R}q'_{!!}p'^{-1}\,\mathrm{K}_{T^*\mathcal{X}} \simeq \mathcal{L}_\sigma\big(T^*\mathcal{Y}, T^*\mathcal{Z}\times T^*\mathcal{X}\big),$$

where $T^*\mathcal{Y}$ is embedded into $T^*\mathcal{Z}\times T^*\mathcal{X}$ by (q,p) and the section σ is given by

$$\sigma = (\omega_X, \omega_Z, -\omega_X, -\omega_Y, -\omega_Y, -\omega_Z).$$

In order to see (2.13) under the non-characteristic hypothesis, it is enough to show that

$$\begin{aligned}&\mathrm{K}_{T^*\mathcal{X}_1}\underset{T^*Y}{\circ}\mathrm{K}_{T^*\mathcal{X}_2} \to \mathcal{L}_\sigma\big(T^*\mathcal{Z}\underset{T^*\mathcal{Z}}{\times}T^*\mathcal{Y}, T^*\mathcal{Z}\times T^*\mathcal{X}\big) \text{ is an isomorphism on } T^*\mathcal{Z}\times\\ &\big(T^*X\times\dot{T}^*(Y\times Y)\times T^*Z\big)\subset T^*\mathcal{Z}\times T^*\mathcal{X}.\end{aligned} \tag{2.14}$$

However it is a consequence of Proposition 1.3.12 (note that (iii) and (v) in the proposition fail on $T^*X\times T^*_YY\times T^*Z$). $\square$

2.6. A vanishing theorem for microlocal holomorphic functions.

Theorem 2.6.1. *Let X be a complex manifold of dimension n. Then, $\mu_X(\mathcal{O}_X)|_{\dot{T}^*X}$ is concentrated in degree $-n$.*

Proof. We may assume $X = \mathbb{C}^n$. Let $q_1 \colon T^*X \times X \to T^*X$ and $q_2 \colon T^*X \times X \to X$ be the projections. Let $p = (x_0, \xi_0) \in \dot{T}^*X$. Then, we have

$$\mu_X(\mathcal{O}_X) \otimes \widetilde{\mathbb{C}}_p \simeq \widetilde{\mathbb{C}}_p \otimes \mathrm{R}q_{1!!}\bigl(\text{“}\varinjlim_{\varepsilon,\delta>0}\text{''}(\mathbb{C}_{\mathrm{F}_{\delta,\varepsilon}} \otimes q_2^{-1}\mathcal{O}_X)\bigr)[2n],$$

where $F_{\delta,\varepsilon} = \bigl\{\bigl((x,\xi),x'\bigr); \delta \geqslant \langle \xi_0, x'-x\rangle > \varepsilon|x'-x|\bigr\}$. Hence it is enough to show that

$$\mathrm{R}q_{1!}(\mathbb{C}_{\mathrm{F}_{\delta,\varepsilon}} \otimes q_2^{-1}\mathcal{O}_X)$$

is concentrated in degree n. We have

$$\mathrm{R}q_{1!}(\mathbb{C}_{\mathrm{F}_{\delta,\varepsilon}} \otimes q_2^{-1}\mathcal{O}_X)_{(x_1,\xi_1)} \simeq \mathrm{R}\Gamma_c\left(\{x' \in X; \delta \geqslant \langle \xi_0, x'-x_1\rangle > \varepsilon|x'-x_1|\}, \mathcal{O}_X\right).$$

The cohomology with compact support of $\mathcal{O}_X$ on the difference of two convex open subsets is concentrated in degree n. □

Now, $H^{-n}\bigl(\mu_X(\mathcal{O}_X)|_{\dot{T}^*X}\bigr)$ has a structure of $\mathcal{E}_X|_{\dot{T}^*X}$-module, i.e. there exists a canonical ring homomorphism $\mathcal{E}_X|_{\dot{T}^*X} \to \mathcal{E}nd\bigl(H^{-n}(\mu_X(\mathcal{O}_X)|_{\dot{T}^*X})\bigr)$.

Indeed, let $p_k \colon X \times X \to X$ be the k-th projection, and $\mathcal{O}^{(0,n)}_{X\times X} := \mathcal{O}_{X\times X} \otimes_{p_2^{-1}\mathcal{O}_X} p_2^{-1}\mathcal{O}^{(n)}_X$. We have morphisms $\mathrm{R}p_{1!}(\mathcal{O}^{(0,n)}_{X\times X}[n] \otimes p_2^{-1}\mathcal{O}_X) \to \mathrm{R}p_{1!}(\mathcal{O}^{(0,n)}_{X\times X}[n]) \to \mathcal{O}_X$ which induce $\mathcal{O}^{(0,n)}_{X\times X}[n] \to \mathrm{R}\mathcal{H}om(p_2^{-1}\mathcal{O}_X, p_1^!\mathcal{O}_X)$. Thus we obtain

$$\begin{aligned}\mathcal{E}_X \to \mu_{\Delta_X}(\mathcal{O}^{(0,n)}_{X\times X}[n]) &\to \mu_{\Delta_X}\bigl(\mathrm{R}\mathcal{H}om(p_2^{-1}\mathcal{O}_X, p_1^!\mathcal{O}_X)\bigr)\\ &\simeq \mu hom(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathrm{R}\mathcal{H}om\bigl(\mu_X(\mathcal{O}_X), \mu_X(\mathcal{O}_X)\bigr).\end{aligned}$$

Hence, Theorem 2.6.1 implies that $\mu_X(\mathcal{O}_X)|_{\dot{T}^*X}$ belongs to $\mathrm{D^b}\bigl(\mathrm{Mod}\bigl(\mathcal{E}_X|_{\dot{T}^*X}, \mathrm{I}(\mathbb{C}_{\dot{T}^*X})\bigr)\bigr)$, the derived category of the abelian category $\mathrm{Mod}\bigl(\mathcal{E}_X|_{\dot{T}^*X}, \mathrm{I}(\mathbb{C}_{\dot{T}^*X})\bigr)$ of ind-sheaves $\mathcal{F}$ on $\dot{T}^*X$ endowed with a ring homomorphism $\mathcal{E}_X|_{\dot{T}^*X} \to \mathcal{E}nd\,(\mathcal{F})$. This implies the following theorem.

Theorem 2.6.2. *Let X be a complex manifold. Then $\mathcal{F} \mapsto \mu hom(\mathcal{F}, \mathcal{O}_X)|_{\dot{T}^*X}$ is a well defined functor from $\mathrm{D^b}(\mathbb{C}_X)$ to $\mathrm{D^b}(\mathcal{E}_X|_{\dot{T}^*X})$.*

References

[K] M. Kashiwara, *Quantization of contact manifold*, Publ. of Research Institute for Mathematical Sciences, **32** no.1 (1996) 1–7.

[KS1] M. Kashiwara and P. Schapira, *Ind-Sheaves*, Astérisque **271**, Soc. Math. France, 2001.

[KS2] ———, *Sheaves on Manifolds*, Springer, 1990.

[KS3] ———, *Microlocal Study of Sheaves*, Astérisque **128**, Soc. Math. France, 1985.

[KS4] ———, *Microlocal Study of Ind-Sheaves I: Micro-Support and Regularity*, Astérisque **284**, Soc. Math. France, 2003.

[KS5] ———, *Ind-sheaves, distributions, and microlocalization*, Sem Ec. Polytechnique, May 18, 1999, arXiv:math.AG/9906200.

[S] M. Sato, *Hyperfunctions and partial differential equations*, in Proc. Int. Conf. on Functional Analysis and related topics, Tokyo Univ. Press, 1969.

[SKK] M. Sato, T. Kawai, and M. Kashiwara, *Hyperfunctions and pseudo-differential equations,* in Komatsu (ed.), *Hyperfunctions and pseudo-differential equations,* Proceedings Katata 1971, Lecture Notes in Mathematics Springer **287** pp. 265–529 1973.

[W] I. Waschkies, *Microlocal perverse sheaves*, to appear in Bull. de la SMF.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606–8502, Japan

E-mail address: masaki@kurims.kyoto-u.ac.jp

Université Pierre et Marie Curie, Institut de Mathématiques, 175 rue du Chevaleret, 75013 Paris, France

E-mail address: schapira@math.jussieu.fr

Université Pierre et Marie Curie, Institut de Mathématiques, 175 rue du Chevaleret, 75013 Paris, France

E-mail address: fivorra@math.jussieu.fr

Université de Nice - Sophia Antipolis Laboratoire J.A. Dieudonne Parc Valrose 06108 Nice, France

E-mail address: ingo@math.nice.fr